THE
MATHEMATICAL GAZETTE.

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SUGGESTIONS FOR THE PRACTICAL TREATMENT OF THE STANDARD CUBIC EQUATION, AND A CONTRIBUTION TO THE THEORY OF SUBSTITUTION.

BY PROF. R. W. GENESE, M.A.

THE graphic methods in use are open to the objection of requiring an improvised graph, depending on the constants of the equation. To obtain even a 5 % approximation, the construction of such a graph demands calculations far in excess of those involved in Horner's method.

The object of the present paper is to advocate the desirability of either *stereotyped graphs*, or *special* tables. The former would also be most useful for didactic purposes.

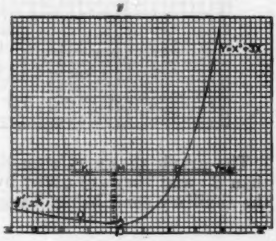
If, in the equation $x^3+3px+q=0$, we put $x=|p|^{\frac{1}{2}}X$,

then $$X^3 \pm 3X = -qp^{-\frac{3}{2}} = c, \text{ say.} \qquad (1)$$

(the sign of the coefficient of X being that of p), and c can be quickly calculated by tables, or slide-rule.

The real value, or values, of X are, then, the abscissae of the intersections of $Y=X^3 \pm 3X$ and $Y=c$.

1°. For $Y=X^3+3X$, the graph shows at once that there is only one real value of X, say X_1.



Unit of x=5 small intervals.
,, $y=\frac{1}{3}$ a small interval.
Solution of $X^3+3X=21$.
$X=2{\cdot}4$ app. and $-1{\cdot}2 \pm 2{\cdot}7i$.

Since the sum of the roots of (1) is zero, if the imaginary roots be $\alpha \pm \beta i$, we have

$$2\alpha + X_1 = 0, \text{ giving } \alpha.$$

And from the coefficient of X,

$$3 = \alpha^2 + \beta^2 + 2\alpha(-2\alpha),$$

$$\text{or} \quad \frac{\beta^2}{3} - \alpha^2 = 1.$$

Hence, if the graph be accompanied by the auxiliary curve

$$\frac{y^2}{3} - x^2 = 1,$$

and the line $2x = -X_1$ meet this curve in Q, Q', *the imaginary roots of the X cubic are represented by OQ, OQ'* (O origin).

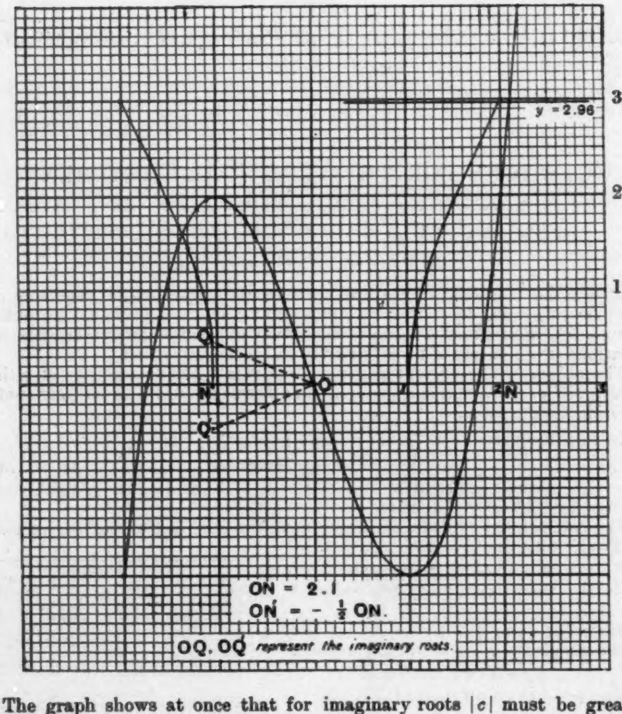
In figure, $MK = -\frac{1}{2}MP$.

Instead of the cubic graph, four-figure tables might be prepared of the values of X^3+3X (for the range likely to be required), from which X_1 might be read off; the conic graph would then give the imaginary roots of X.

Or, with loss of time, if tables of sinh and cosh be available, the substitution $X=2\sinh v$ gives $\sinh 3v=c/2$; v being found, the imaginary roots are $-\sinh v\pm\sqrt{-3}\cosh v$.

2°. $$X^3-3X=c.$$

Graph for $c=2{\cdot}96$.



The graph shows at once that for imaginary roots $|c|$ must be greater than 2: and they are given by a construction similar to the previous one, the auxiliary conic being changed to

$$x^2-\frac{y^2}{3}=1.$$

In studying the relation between the roots (x_1, x_2, x_3) of the cubic equation, the form $x^3-3x+c=0$ is to be preferred, as including the "irreducible case." (The slight modifications of the formulae obtained for this form, consequent on a change to $x^3-3px+c=0$, will be given at the end of this note.)

If we use the transformation

$$y=\alpha x^2+\beta x+\gamma\equiv S(x),$$

the roots of the resulting cubic in y are $S(x_1)$, $S(x_2)$, $S(x_3)$; and, theoretically, we can choose α, β, γ so that the cubic in y is identical with that in x.

Then $S(x_1)=x_1$, x_2 or x_3; and $S(x_2)$, $S(x_3)$ supply the other roots.

There are three possibilities.

1°. $S(x_1)=x_1;\quad S(x_2)=x_2;\quad S(x_3)=x_3;$ ....(A)

but, since S is quadratic, this can only arise when two of the roots are equal, or $|c|=2$.

2°. $S(x_1)=x_1;\quad S(x_2)=x_3;\quad S(x_3)=x_2.$ ....(B)

3°. $S(x_1)=x_2;\quad S(x_2)=x_3;\quad S(x_3)=x_1.$ ....(C)

Distinct formulae will be obtained for each case.

The relation necessary for the vanishing of y^2 gives

$$\gamma=-2\alpha.$$

Thus $$S(x)\equiv\alpha(x^2-2)+\beta x.$$

The next relation, from comparing terms of first degree, is

$$c=\frac{\alpha^2+\beta^2-1}{\alpha\beta}.$$

And the third, $\alpha^3c^2-(3\alpha^2\beta+\beta^3-1)c+\alpha(6\beta^2-2\alpha^2)=0.$

Eliminating c, *in order to obtain the relation between α and β for all cubics*, there results a cubic equation in α^2, which, however, admits of the solutions

$$\alpha^2=(\beta-1)^2,\quad \beta^2+\beta \quad\text{and}\quad \beta^2+\beta+1.$$

1°. $\alpha^2=(\beta-1)^2$ gives $c=\pm2$, that is, applies only to the case of equal roots. Taking $\alpha=\beta-1$, $c=2$,

$$\begin{aligned}S(x)&\equiv\alpha(x^2-2+x)+x\\&=x \text{ when } x=+1 \text{ or } -2.\end{aligned}$$

With $\alpha=1-\beta$, $c=-2$, and

$$\begin{aligned}S(x)&\equiv\alpha(x^2-2-x)+x\\&=x \text{ when } x=-1 \text{ or } +2.\end{aligned}$$ This is the case (A).

2°. $\alpha^2=\beta^2+\beta$, and then

$$\begin{aligned}c=\frac{2\beta^2+\beta-1}{\beta^{\frac{3}{2}}(\beta+1)^{\frac{1}{2}}}&=-\frac{1-2\beta}{\beta}\,\frac{(1+\beta)^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\\&=-\left(\frac{1}{\beta}+1-3\right)\left(\frac{1}{\beta}+1\right)^{\frac{1}{2}}\end{aligned}$$

and, therefore, one root x_1 of the cubic is $\left(\frac{1}{\beta}+1\right)^{\frac{1}{2}}=\frac{\alpha}{\beta}$.

$$\begin{aligned}S(x_1)=\alpha(x_1^2-2)+\beta x_1&=\beta x_1(x_1^2-1)\\&=\beta x_1\left(\frac{1}{\beta}\right)\\&=x_1.\end{aligned}$$

Hence, since the equation has not equal roots, we must have

$$S(x_2)=x_3 \text{ and } S(x_3)=x_2;\ \text{which is case (B).}$$

Ex. gr. For $x^3-3x=\frac{9}{8}$, $x_1=\frac{3}{2}$.

$$\begin{aligned}S(x_1)&=\tfrac{6}{5}(x_1^2-2)+\tfrac{4}{5}x_1=\tfrac{1}{5}x_1+\tfrac{4}{5}x_1=x_1,\\S(x_2)&=\tfrac{6}{5}(x_2^2+\tfrac{3}{2}x_2-\tfrac{3}{4})-x_2-\tfrac{3}{2}\quad=x_3.\end{aligned}$$

Conversely, if we begin with the last equations,

$$\alpha(x_2{}^2-2)+\beta x_2=x_3,$$
$$\alpha(x_3{}^2-2)+\beta x_3=x_2,$$

we get, subtracting and dividing by x_2-x_3,

$$\alpha(x_2+x_3)+\beta=-1,$$
$$\text{i.e.}\quad -\alpha x_1+\beta=-1,$$
$$x_1=\frac{1+\beta}{\alpha}$$
$$=\frac{\alpha}{\beta},$$

and, thus, return to the condition $\alpha^2=\beta^2+\beta$.

3°. $$\alpha^2=\beta^2+\beta+1.$$

There is nothing left for this but case (C).

For the equation $x^3-3px+c=0$, the relations become

$$\gamma=-2p\alpha,$$
$$c=\frac{p^2\alpha^2+p\beta^2-p}{\alpha\beta},$$

and $$p\alpha^2=(\beta-1)^2,\quad \beta^2+\beta\quad \text{and}\quad \beta^2+\beta+1$$

in the cases (A), (B), (C). $\left(\text{The reproduced root in (B) is } \frac{p\alpha}{\beta}\right)$.

It may be of interest to add that to every transformation of the form

$$y=\alpha(x^2-2p)+\beta x,$$

there corresponds *only one cubic equation*. *Ex. gr.* $y=x^2-14+2x$ gives

$$p=7\ (=2^2+2+1),\qquad c=\frac{49+28-7}{2}=35,$$

and the equation is $$x^3-21x+35=0.$$

All the above results may also be obtained from the sets of S equations (A), (B), (C).

Any values may be assigned to α and β. Then the last relation gives p and the preceding c.

The theory may be restated thus:

The condition that $y=\alpha x^2+\beta x+\gamma$ should be a reproducing transformation, for some cubic $x^3+3px+q=0$ is

$$-\tfrac{1}{2}\alpha\gamma=\text{either } (\beta-1)^2\quad\text{or}\quad \beta^2+\beta\quad\text{or}\quad \beta^2+\beta+1.$$

In the 1st case, each root is untransformed.

" 2nd " one root $\left(=\frac{p\alpha}{\beta}\right)$ is untransformed,
the other two are interchanged.

" 3rd " the roots are transformed cyclically.

To each transformation corresponds *only one* cubic equation, given by

$$p=-\frac{\gamma}{2\alpha},$$
$$q=\frac{p^2\alpha^2+p\beta^2-p}{\alpha\beta}.$$

Example of the 2nd case (α, β are at our disposal).

Let $\alpha=1$, $\beta=2$, then $\gamma=-12$, $p=6$, $q=27$.

Thus, to $y=x^2+2x-12$ there corresponds

$$x^3-18x+27=0=(x-3)(x^2+3x-9),$$
$$S(3)=9+6-12=3.$$

If x_2, x_3 be the other roots,

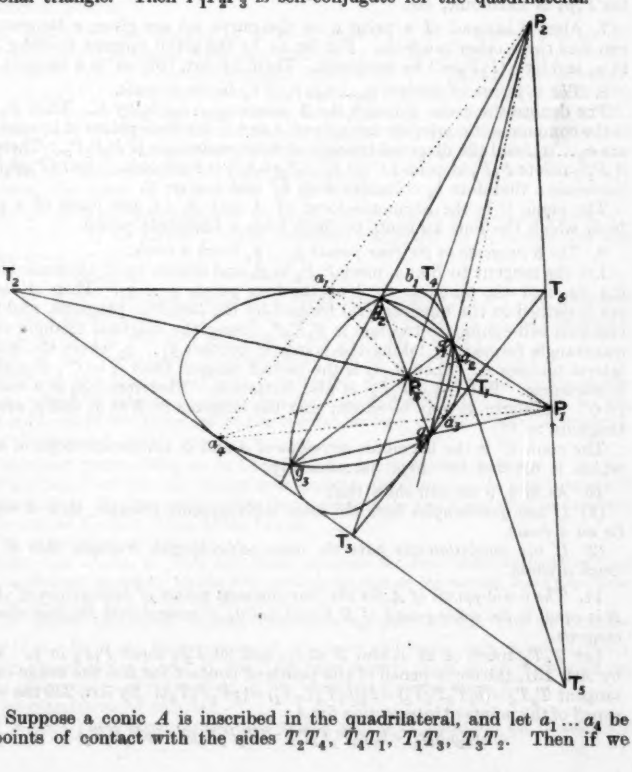
$$S(x_2)=(9-3x_2)+2x_2-12=-3-x_2=x_3, \because x_2+x_3=-3.$$

Aberystwyth, August 1916.

8-POINT AND 8-TANGENT CONICS, AND AN ASSOCIATED PENCIL AND RANGE OF CONICS.

The following notes are a development of subject-matter which is treated in the writer's book on *Cross-Ratio Geometry*, chiefly in Chap. XIV., and the reader is referred to articles in that work for the demonstration of any property that is given here without proof.

1. In the figure, $T_1 \dots T_6$ is a complete quadrilateral, $P_1P_2P_3$ being its diagonal triangle. Then $P_1P_2P_3$ is self-conjugate for the quadrilateral.



2. Suppose a conic A is inscribed in the quadrilateral, and let $a_1 \dots a_4$ be its points of contact with the sides T_2T_4, T_4T_1, T_1T_3, T_3T_2. Then if we

take any one of them as a_1, the others can be found by means of the ruler by joining a_1 to the points P_1, P_2, P_3, since $P_1P_2P_3$ is a self-conjugate triangle for A (Art. 211). Suppose a second conic B is inscribed in the quadrilateral, and let $b_1 \dots b_4$ be its points of contact as above. Then from any one of the b's the other three can be found by the ruler, and $P_1P_2P_3$ is the common self-conjugate triangle of A and B.

3. Let A and B intersect in the four points $g_1 \dots g_4$. Then if one of the g's is given, the other three can be found by the ruler, and $P_1P_2P_3$ is self-conjugate for the quadrangle $g_1 \dots g_4$.

4. By Art. 190, $P_3(a_1a_2T_2T_4)$ is harmonic, as is also $P_3(b_1b_2T_2T_4)$. Therefore P_3P_1, P_3P_2 are the double rays of the involution pencil whose characteristic is $P_3(a_1a_2,\ b_1b_2)$.

5. By Art. 213, Cor., $P_3(a_1b_1g_1g_2)$ is harmonic, as is also $P_3(a_2b_2g_1g_2)$. Therefore the common chords g_1g_3, g_2g_4 are the double rays of the involution pencil $P_3(a_1b_1,\ a_2b_2)$.

6. From the above we infer that if we are given a triangle which is self-conjugate for a conic, and one point a on the curve, we can find three other points on the conic. For if aP_1 meets P_2P_3 in p, and the curve again in a', $(aa'P_1p)$ is harmonic, etc.

7. Also, if instead of a point a on the curve we are given a tangent, we can find three other tangents. For let $a\alpha$ be the given tangent meeting P_1P_3 in α, and let $\alpha(P_1P_2aa')$ be harmonic. Then, by Art. 162, $\alpha a'$ is a tangent, etc.

8. *The 8 points of contact $a_1 \dots a_4$, $b_1 \dots b_4$ lie on a conic.*

For denote the conic through the 5 points $a_1 \dots a_4$, b_1 by C. Then $P_1P_2P_3$ is the common self-conjugate triangle of A and C, for their points of intersection are $a_1 \dots a_4$, and the diagonal triangle of this quadrangle is $P_1P_2P_3$. Therefore if P_2b_1 meets P_1P_3 in p and C in b_1', $(P_2pb_1b_1')$ is harmonic. And $(P_2pb_1b_4)$ is harmonic; therefore b_4 coincides with b_1' and lies on C.

The conic C is the harmonic locus of A and B, *i.e.* the locus of a point from which the four tangents to them form a harmonic pencil.

9. *The 8 tangents at the four points $g_1 \dots g_4$ touch a conic.*

Let the tangent to B at g_1 meet P_1P_3 in β, and denote by C' the conic touching βg_1 and the tangents to A at the four points $g_1 \dots g_4$. Then A and C' are inscribed in the quadrilateral formed by the last four tangents, and their common self-conjugate triangle is $P_1P_2P_3$, being the diagonal triangle of the quadrangle formed by taking the points of contact $g_1 \dots g_4$ where the quadrilateral touches A. Then if βg is the second tangent from β to C', $\beta(g_1gP_2P_3)$ is harmonic. But $\beta(g_1g_4P_2P_3)$ is also harmonic. Therefore βg_4 is a tangent to C'. Similarly, it may be shown that the tangents to B at g_2 and g_3 are also tangents to C'.

The conic C' is the harmonic envelope of A and B, *i.e.* the envelope of a line which is divided by them harmonically.

10. As in 8, 9 we can show that

(1) *If two quadrangles have the same self-conjugate triangle, their 8 vertices lie on a conic.*

(2) *If two quadrilaterals have the same self-conjugate triangle, then 8 sides touch a conic.*

11. *The conic-pencil of A for the four common points of intersection of A and B is equal to the conic-pencil of B for its points of contact with the four common tangents.*

Let T_2T_4 touch A at a_1 and B at b_1, and let P_2b_1 meet P_1P_3 in p. Then, by Art. 131, the conic-pencil of the points of contact for $B =$ the range on the tangent $T_2T_4 = (b_1T_4T_6T_2) = P_2(b_1T_4T_6T_2) = (pP_3P_1T_2)$. By Art. 239 the conic-pencil of the points of intersection for A

$$= a_1(g_2g_1g_4g_3) = b_1(a_1P_1P_3P_2) = (T_2P_1P_3p) = (pP_3P_1T_2).$$

12. Now, the two conics A and B taken together may be considered either as the base of a pencil of conics circumscribing the quadrangle $g_1 \dots g_4$, or of a range of conics inscribed in the quadrilateral $T_1 \dots T_4$, and both the pencil and the range have $P_1P_2P_3$ for their common self-conjugate triangle. In what follows, when a pencil or range is mentioned, the reference is always to that determined by A and B.

13. From 10, it at once follows that

(1) *Given four conics of a range, the four common points of intersection of one pair and the four common points of intersection of the other pair are* 8 *points on a conic.*

(2) *Given four conics of a pencil, the four common tangents of one pair and the four common tangents of the other pair are* 8 *tangents of a conic.*

14. Let any conic M of the pencil cut T_2T_4 in the point m_1. Then, if we consider that conic of the range which touches T_2T_4 at m_1, and if m_2, m_3, m_4 are its points of contact with T_4T_1, T_1T_3, T_3T_2, M will pass through these points.

Also, if μ_1 is the second point in which M cuts T_2T_4, the same properties will hold for the points μ_2, μ_3, μ_4, and by Desargues' Theorem, Art. 187, $(a_1b_1m_1\mu_1)$ is harmonic, so that if m_1 is given, the conic M will pass through 12 known points.

15. Again, let the tangent to M at g_1 meet P_1P_3 in γ, and denote by M' the range-conic which touches γg_1, and let γg be the second tangent to M'. Then, since $P_1P_2P_3$ is a self-conjugate triangle of M', $\gamma(g_1gP_2P_3)$ is harmonic. But $\gamma(g_1g_4P_2P_3)$ is also harmonic. Therefore γg_4 is a tangent to M'. Similarly, it may be shown that the tangents to M at g_2 and g_3 are also tangents to M'.

Also, if $g_1\gamma'$ is the second tangent from g_1 to M', the same properties will hold for the second tangents to M' from the points g_2, g_3, g_4, and by the correlative of Desargues, Art. 188, $g_1(\gamma\gamma' T_1T_2)$ is harmonic, so that if $g_1\gamma$ is given, the conic M' will touch 12 given lines.

16. *A, B, C are three conics of a pencil, and are cut by a transversal in a, a'; b, b': c, c'. If the points c, c' are harmonic conjugates for a, b, they will also be the same for a', b'.*

For by Desargues, $(a'b'c'c) = (abcc') = -1$.

17. *A, B, C' are three conics of a range, and from a point P are drawn to them pairs of tangents Pa, Pa'; Pb, Pb'; Pc, Pc'. If Pc, Pc' are harmonic conjugates for Pa, Pb, they will also be the same for Pa', Pb'.*

For by the correlative of Desargues', $P(a'b'c'c) = P(abcc') = -1$.

18. Now, if a transversal cuts two circles, and tangents are drawn at the points where it intersects them, the four points where the tangents to the first circle meet the tangents to the second lie on a coaxial circle (Townsend, *Mod. Geom.*, Art. 194).

Therefore, generalising as in Chap. XIX., we have the property:

If a transversal cuts two conics, and tangents are drawn at the points where it intersects them, the four points where the tangents to the first conic meet the tangents to the second lie on a conic of the pencil.

The correlative of this is:

Given two conics, if from any point we draw the four tangents to them, and join each point of contact on the first conic to the two on the second, these four lines will be tangents to a conic of the range.

The above principles suggest the following propositions, due to Chasles.

1. If A, B, C are three conics of a pencil, and from each point of C are drawn tangents to A and B, the four lines joining their point of contact on A to their points of contact on B will envelop a range conic C'.

The correlative of this is:

If A, B, C' are three conics of a range, and a tangent roll on C' meeting

A in a, a', and B in b, b', the four points where the tangents to A at a, a' intersect the tangents to B at b, b' lie on a conic C of the pencil.

2. A, B, C are three conics of a pencil, and on A, B are taken two points a, b which are conjugate for C, and the tangents at a and b meet in P. Then

(1) The locus of P is a conic of the pencil.

(2) The envelope of the line ab is that conic of the range which touches the four tangents to C drawn at the common points of the pencil.

Its correlative is:

A, B, C' are three conics of a range, and to A and B are drawn two tangents Pa, Pb which are conjugate for C'. Then

(1) The envelope of the line ab is a conic of the range.

(2) The locus of P is that conic of the pencil which passes through the point where C' touches the common tangent of the range. JOHN J. MILNE.

APPROXIMATIONS TO $\sqrt[n]{1+x}$, WHERE n IS AN INTEGER AND $0<x<1$.

(*Concluded.*)

By J. M. CHILD.

CONTINUED FRACTION APPROXIMATIONS. II.

Convergents to $\tan n\theta$.

If the numerator and denominator of the fraction are both convergent series, one value of $\tan n\theta$ is

$$\frac{c_1 \tan\theta - c_3 \tan^3\theta + \dots}{1 - c_2 \tan\theta + c_4 \tan^4\theta - \dots},$$

where $(1+x)^n \equiv 1 + c_1 x + c_2 x^2 + \dots$, and n is any number.

This fraction can be converted into a continued fraction, and we have

$$\tan n\theta = \frac{n\tan\theta}{1-}\,\frac{(n^2-1)\tan^2\theta}{3-}\,\frac{(n^2-2^2)\tan^2\theta}{5-}\,\frac{(n^2-3^2)\tan^2\theta}{7-}\dots.$$

If n is an integer, positive or negative, the continued fraction terminates; thus we have

$$\tan 2\theta = \frac{2\tan\theta}{1-}\,\frac{(2^2-1)\tan^2\theta}{3} = \frac{2\tan\theta}{1-\tan^2\theta},$$

$$\tan(-3\theta) = \frac{-3\tan\theta}{1-}\,\frac{(3^2-2)\tan^2\theta}{3-}\,\frac{(3^2-2^2)\tan^2\theta}{5} = -\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta},$$

and so on.

If n is a fraction $=\dfrac{1}{m}$ say, where m is any number greater than unity, the C.F. becomes (where $t=\tan\theta$)

$$\tan\frac{\theta}{m} = \frac{t}{m+}\,\frac{(m^2-1)t^2}{3m+}\,\frac{(2^2m^2-1)t^2}{5m+}\,\frac{(3^2m^2-1)t^2}{7m+}\dots,$$

and this would appear to be convergent, by the test $b_{n-1}.b_n > a_n$, if

$$(2r+1)(2r+3)m^2 > (r^2m^2-1)t^2, \quad \textit{i.e.} \text{ if } t<2.$$

Between 1 and 2, however, the convergence is slow, and it is better to use $t=1$ as an upper limit.

Thus, to find $\tan 15°$, take $t=1$, $m=3$, and we have

$$\tan 15° = \frac{1}{3+}\,\frac{8}{9+}\,\frac{35}{15+}\,\frac{80}{21+}\,\frac{143}{27+}\,\frac{224}{33+}\,\frac{323}{39+}\,\frac{440}{45+}\,\frac{575}{51+},$$

and the successive convergents are

$$\frac{1}{3},\ \frac{9}{35},\ \frac{170}{630},\ \frac{4290}{16030},\ \frac{140140}{522900},\ \frac{5585580}{20846420},\ \frac{263102840}{981907080},\ \ldots.$$

The last convergent gives a value 0·2679508, known to be in excess, which agrees with $2-\sqrt{3}=0·2679492$.

Again, to find $\tan(\frac{1}{3}\tan^{-1}2)$, we have the continued fraction

$$\frac{1}{10+}\ \frac{12}{15+}\ \frac{99}{100+}\ \frac{224}{25+}\ \frac{399}{180+}\ \frac{624}{55+},$$

of which the sixth convergent (in defect) $=\dfrac{659426355}{7090894062}$.

This gives $\qquad \tan^{-1}(\frac{1}{3}\tan^{-1}2)=0·09299622(+\ldots)$.

Verification from tables gives

$$\tan^{-1}\tfrac{1}{2}=26°\,33'\,54''·2, \qquad \tan^{-1}0·9299622=5°\,18'\,46''·2,$$

showing an error of less than 1″.

Similarly $\qquad \tanh\dfrac{x}{m}=\dfrac{t}{m-}\ \dfrac{(m^2-1)t^2}{3m-}\ \dfrac{(4m^2-1)t^2}{7m-}\ldots,$

where t is now $\tanh x$.

We will now consider the general approximations for particular values of m.

I. **m = 3. Trisection of an Angle.**

We have $\qquad \tan\dfrac{\theta}{3}=\dfrac{t}{3+}\ \dfrac{8t^2}{9+}\ \dfrac{35t^2}{15+}\ \dfrac{80t^2}{21+}\ \dfrac{143t^2}{27+}\ \dfrac{224t^2}{33+}\ldots.$

The successive convergents to this continued fraction are

$$\frac{t}{3},\ \frac{9t}{27+8t^2},\ \frac{135t+35t^3}{405+225t^2},\ \frac{2835t+1455t^3}{8505+6885t^2+640t^4},\ \frac{76545t+58590t^3+5005t^4}{229635+143810t^2+32175t^4},\ \ldots.$$

The corresponding limits of error are

$$\frac{8t^3}{3(27+8t^2)},\ \frac{280t^5}{(27+8t^2)(405+225t^2)},\ \frac{22400t^7}{(405+225t^2)(8505+6885t+640t^4)},\ \ldots,$$

which are less than, for particular values of t,

$$\frac{1}{13},\ \frac{1}{78},\ \frac{1}{900},\ \ldots, \text{ when } t=1\ (45°);$$

$$\frac{1}{87},\ \frac{1}{1500},\ \frac{1}{27000}\ \ldots, \text{ when } t=\tfrac{1}{2}\ (25°).$$

Hence for an angle less than 45° the third approximation, which is equal to

$$\frac{t}{3}\,\frac{1+\frac{7}{27}t^2}{1+\frac{5}{9}t^2},$$

yields with a unit anything less than 10 inches an error less than 0·01 inch about: whilst the error for an angle of less than 25° will not be given in practice with anything less than a unit of 22 feet.

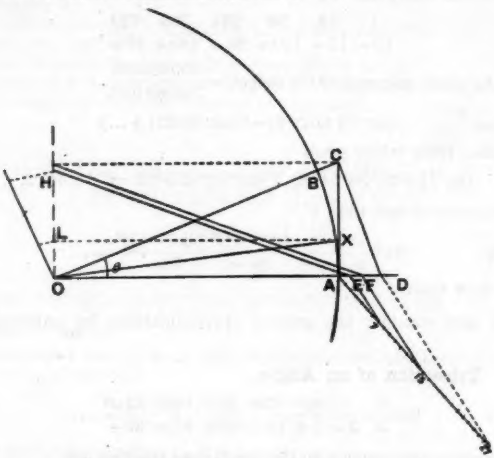
Hence the following construction gives the trisection of the angle with an error less than the ordinary error in drawing.

Let AOC be the angle less than 25°, $AC\perp OA$; draw $CD\perp OC$ to cut OA produced in D; find AE, AF respectively $\frac{7}{27}$ and $\frac{5}{9}$ of AD: construct OH a fourth proportional to AC, AE, AF: find AX equal to third of OL.

Then $$AX=\frac{AC}{3}\cdot\frac{OE}{OF},$$

i.e. $$\tan XOA=\frac{\tan\theta}{3}\cdot\frac{1+\frac{7}{27}\tan^2\theta}{1+\frac{5}{9}\tan^2\theta}=\tan\theta/3 \text{ (approx.)}.$$

For larger angles cut off three equal arcs and trisect the remainder, supposed to be left less than 25° of arc.



II. Cube Root of the Complex $\alpha+\beta\iota$.

We have $\sqrt[3]{\alpha+\beta\iota}=r^{\frac{1}{3}}(\cos\theta/3+\iota\sin\theta/3)$ as one value, where $r^2=\alpha^2+\beta^2$ and $\tan\theta=\beta/\alpha$.

If $$\alpha\geqq\beta,\quad \frac{\beta}{\alpha}\leqq 1,\quad \tan\theta/3=\frac{\beta}{3\alpha+}\,\frac{8\beta^2}{9\alpha+}\,\frac{35\beta^2}{15\alpha+}\dots.$$

If $\alpha<\beta$, $\dfrac{\beta}{\alpha}>1$, we can take $\tan(90^\circ-\theta)=\dfrac{\alpha}{\beta}$, and find $\tan(30^\circ-\theta/3)$ and thence $\theta/3$: the sixth root of $\alpha^2+\beta^2$ can then be found by the Binomial Theorem, or the approximation $\dfrac{2n+\overline{n+1}x}{2n+(n-1)x}$, or higher convergents to

$$\sqrt[n]{1+x}=1+\frac{x}{n+}\,\frac{(n-1)x}{2+}\,\frac{(n+1)x}{3n+}\,\frac{(2n-1)x}{2+}\dots,$$

when these approximations are easily applicable; or else by logarithms.

Ex. 1. To find $\sqrt[3]{2+\sqrt{3}\iota}=7^{\frac{1}{6}}(\cos\theta/3+\sin\theta/3)$, where $\tan\theta=\dfrac{\sqrt{3}}{2}$, we have

$$\tan\theta/3=\sqrt{3}\left(\frac{1}{6+}\,\frac{24}{18+}\,\frac{105}{30+}\,\frac{240}{42+}\,\frac{529}{54+}\,\frac{672}{66+}\dots\right),$$

and the fifth and sixth convergents give 0·140018 and 0·140006 respectively;

$$\therefore\ \tan\theta/3=0{\cdot}24251;$$

$$\therefore\ \cos\theta/3=0{\cdot}97185 \quad\text{and}\quad \sin\theta/3=0{\cdot}23568.$$

Also, $\sqrt[3]{7}=2\sqrt[3]{1-\frac{1}{8}}=1{\cdot}91293$; $\therefore\ \sqrt[6]{7}=1{\cdot}38308$.

Hence $\sqrt[3]{2+\sqrt{3}\iota}=1{\cdot}34915+0{\cdot}32596\iota$.

III. Solution of the Cubic Equation.

Let $x^3-3px-2r=0$ be the equation to be solved: one root, by Cardan's method, is

$$x=(r+\sqrt{r^2-p^3})^{\frac{1}{3}}+(r-\sqrt{r^2-p^3})^{\frac{1}{3}}.$$

If $p^3>r^2$ the method fails, but a root is given as $x=2\sqrt{p}\cos\theta/3$, where $\tan\theta=\frac{\sqrt{p^3-r^2}}{r}=\frac{\sqrt{q}}{r}$, say.

Hence *a general algebraic* solution in the "irreducible case" is

$$x=\frac{2\sqrt{p}}{\sqrt{1+t^2}},\quad \text{where } t=\sqrt{q}\left(\frac{1}{3r+}\,\frac{8q}{9r+}\,\frac{35q}{15r+}\cdots\right),\quad \text{where } q=p^3-r^2.$$

Note that the other, Cardan's, method can be replaced by a method where a tanh takes the place of the tan.

It will be found that this method of solution is not notable for shortness, but it is put forward as the *general algebraic solution of a cubic in terms which do not involve imaginary quantities or reference to trigonometrical tables.* It is claimed that an infinite convergent C.F. is *superior* to a radical as a solution if the convergence is rapid.

Ex. 1. Solve $\qquad x^3-7x-6=0.$

Here $\qquad \sqrt{p}=\sqrt{\tfrac{7}{3}},\quad r=3\quad \text{and}\quad \sqrt{q}=\sqrt{\tfrac{100}{27}}.$

Hence a root is $2\sqrt{\tfrac{7}{3}/(1+t^2)}$, where

$$t=10\sqrt{3}\left\{\frac{1}{81+}\,\frac{2400}{243+}\,\frac{10500}{405+}\,\frac{24000}{567+}\,\frac{42900}{729+}\cdots\right\}.$$

The fourth and fifth convergents to the C.F. are

0·011110302 (in defect) and 0·0111111181 (in excess).

Taking 1/90 as an approximation, we obtain

$$t=10\sqrt{3}/90,\ \text{and the root}=2\sqrt{\tfrac{7}{3}/(1+\tfrac{1}{27})}=3.$$

The angle "θ" can always be made smaller than $\tan^{-1}\frac{1}{2}$ by either one or other of the 'dodges':

(i) If $\tan^{-1}\frac{1}{2}<\theta<\tan^{-1}1$; use $\phi=45^\circ-\theta$, $\theta/3=\tan^{-1}(2-\sqrt{3})-\phi/3$.

(ii) If $\tan^{-1}1<\theta<\tan^{-1}2$; use $\phi=\theta-45^\circ$, $\theta/3=\tan^{-1}(2-\sqrt{3})+\phi/3$.

(iii) If $\tan^{-1}2<\theta<\tan^{-1}\infty$; use $\phi=90^\circ-\theta$, $\theta/3=\tan^{-1}\frac{\sqrt{3}}{3}-\phi/3$.

Ex. 1. Solve $\qquad x^3-3x-1=0.$

Here $\qquad r=\tfrac{1}{2},\quad \sqrt{p}=1,\quad \sqrt{q}=\frac{\sqrt{3}}{2},\quad \tan\theta=\sqrt{3}.$

Take $\qquad \tan\phi=\tan(\theta-45^\circ)=\frac{\sqrt{3}-1}{\sqrt{2}+1}=2-\sqrt{3}\ (=\tfrac{1}{4}\text{ about}).$

Now $\tan\phi/3$ is given by the approximation

$$\frac{t}{3}\cdot\frac{27+7t^2}{27+15t^2},\ \text{where } t=\tan\phi,$$

with an error in excess $<\frac{22400\,.\,4^{-7}}{419\,.\,8937}$ or $4\,.\,10^{-7}$ when $t=\frac{1}{4}$;

i.e. $$\tan\frac{\phi}{3}=\frac{2-\sqrt{3}}{3}\cdot\frac{27+7(7-4\sqrt{3})}{27+15(7-4\sqrt{3})}=0{\cdot}0874892,$$

is correct to six decimal places;

$$\therefore\ \tan\frac{\theta}{3}=\frac{2-\sqrt{3}+0{\cdot}087489}{1-(2-\sqrt{3})(0{\cdot}087489)}=0{\cdot}363970.$$

Hence a root is $2\Big/\sqrt{1+\tan^2\frac{\theta}{3}}=1{\cdot}879386.$

Ex. 2. Solve $$x^3-6x-2=0.$$

Here $$p=2,\quad r=1 \text{ and } \sqrt{q}=\sqrt{7}:\ \tan\theta=\sqrt{7}.$$

Take $$\tan\phi=\tan(90^\circ-\theta)=\frac{1}{\sqrt{7}}\ (=\tfrac{3}{8}\text{ about});$$

then $$\tan\frac{\phi}{3}=\frac{\sqrt{7}}{21}\,\frac{27+1}{27+\frac{1}{7}^{5}}=\frac{7\sqrt{7}}{153}=0{\cdot}121047,$$

where the error in excess is nearly, but less than, 8.10^{-6}:

$$\therefore\ \tan\theta/3=\frac{\frac{\sqrt{3}}{3}-0{\cdot}121047}{1+\frac{\sqrt{3}}{3}\times(0{\cdot}121047)}=0{\cdot}42650;$$

in this the error is now in defect and slightly smaller, hence the result given is probably correct to the last figure.

Here a root is $2\sqrt{2\Big/\left(1+\tan^2\frac{\theta}{3}\right)}=2{\cdot}6017.$

IV. Solution of the Biquadratic $x^4-3px^2+qx+r=0$.

If e^2-2p is a root of the equation
$$z^3-(3p^2+4r)z+2p^3-8pr-q^2=0,$$
i.e. if $$e^2=2[p+\sqrt{P/(1+T^2)}],$$
where $$T=\sqrt{Q}\left\{\frac{1}{3R+}\,\frac{8Q}{9R+}\,\frac{35Q}{15R+}\cdots\right\},$$
$$P=p^2+\frac{4r}{3},\quad R=-\left(p^3-4pr-\frac{q^2}{2}\right),$$
and $$Q=12p^4r-\frac{8}{3}p^2r^2+\frac{64r^3}{27}+p^3q^2-4prq^2-\frac{q^4}{4},$$
then the roots of the biquadratic equation are
$$\tfrac{1}{2}[e\pm\sqrt{6p-2q/e-e^2}]\quad\text{and}\quad\tfrac{1}{2}[-e\pm\sqrt{6p+2q/e-e^2}].$$

Ex. 1. Solve $$x^4-15x^2+10x+24=0.$$

Here $$e^2-10 \text{ is a root of } z^3-171z-810=0$$
or $$\tfrac{1}{3}(e^2-10) \text{ is a root of } z^3-19z-30=0.$$

As it happens the roots of this cubic are obvious on inspection, as will usually be the case when there are integral roots to the biquadratic.

But a root can be found, as before, thus:

The root is $2\sqrt{\frac{19}{3}\Big/(1+t^2)}$, where $\left(\text{since } \frac{\sqrt{q}}{r}=\frac{\sqrt{2472}}{135}=\frac{3}{8}\text{ nearly}\right)$
$$t=\sqrt{2472}\left\{\frac{1}{405+}\,\frac{19776}{1215+}\,\frac{79520}{2025+}\cdots\right\},$$
giving an error in excess $<6.10^{-6}$ with the third convergent.

Hence $$t=0{\cdot}11815\text{ nearly};$$
$$\therefore\ \text{root}=4{\cdot}999\text{ (in defect)}=5;$$
$$\therefore\ e^2=25\quad\text{and}\quad e=5,$$
and $$\sqrt{6p-\frac{2q}{e}-e^2}=\sqrt{30-4-25}=1,$$
$$\sqrt{6p+\frac{2q}{e}-e^2}=\sqrt{30+4-25}=3.$$

Therefore the roots of the biquadratic are 2, 3, -1, -4.

Ex. 2. Solve $$x^4+2x^2+x+2=0.$$

Here $e+\frac{4}{3}$ is a root of the cubic equation
$$z^3-\tfrac{28}{3}z+\tfrac{245}{27}=0$$
or $\frac{1}{7}(3e^2+4)$ is a root of $z^3-\frac{12}{7}z+\frac{5}{7}=0$. The root 1 is obvious; but if it had not been so, we should have had the approximate value
$$-2\sqrt{\frac{4}{7}}\Big/\sqrt{1+t^2},\ \text{where } t=\frac{9\sqrt{7}}{35}.$$

This gives $\frac{1}{7}(3e^2+4)$ equal to the negative root $-1{\cdot}48$; the sum of the other pair is therefore $1{\cdot}479$ and the product $\frac{5}{7}\div 1{\cdot}479=0{\cdot}479$, giving 1 and $0{\cdot}479$.

Taking $$\tfrac{1}{7}(3e^2+4)=1,\quad e^2=1,$$
and $$x^4+2x^2+x+2\equiv(x^2+x+1)(x^2-x+2).$$

V. m=5. Division of Angle.

We have $$\tan\frac{\theta}{5}=\frac{t}{5+}\,\frac{24t^2}{15+}\,\frac{99t^2}{25+}\,\frac{224t^2}{35+}\cdots.$$

The third convergent $\dfrac{t}{5}\cdot\dfrac{125+33t^2}{125+73t^2}$ has an error in excess less than 7.10^{-3} if $t=1$, or 4.10^{-5} if $t=\frac{1}{2}$, or 1.10^{-7} if $t=\frac{1}{4}$.

This gives a construction similar to the one for the trisection (with about the same limits of accuracy), the ratios $AE:EF:FD$ being $33:56:86$.

Similar results are obtained for other values of m.

VI. m=∞. Series for tan⁻¹x.

Since $\lim\limits_{m\to\infty} m\tan\theta/m=\theta$,
$$\therefore\ \theta=\lim_{m\to\infty}\frac{t}{1+}\,\frac{\left(1-\frac{1}{m^2}\right)t^2}{3+}\,\frac{\left(2^2-\frac{1}{m^2}\right)t^2}{5+}\,\frac{\left(3^2-\frac{1}{m^2}\right)t^2}{7+}\cdots$$
$$=\frac{t}{1+}\,\frac{1^2t^2}{3+}\,\frac{2^2t^2}{5+}\,\frac{3^2t^2}{7+}\cdots,$$
i.e. $$\tan^{-1}x=\frac{x}{1+}\,\frac{x^2}{3+}\,\frac{4x^2}{5+}\,\frac{9x^2}{7+}\cdots,$$
and this is convergent if $t(r-1)<\sqrt{4r^2-1}$ for all integral values of r, which certainly is the case if $t<1$.

VII. m=0. Series used by Lambert.

Let m and θ tend to zero, so that $\dfrac{\theta}{m}$ is constant $=\alpha$; then, since
$$\tan\frac{\theta}{m}=\frac{t}{m}\cdot\frac{1}{1+}\,\frac{t^2-\frac{t^2}{m^2}}{3+}\,\frac{4t^2-\frac{t^2}{m^2}}{5+}\cdots$$
and $$\frac{t}{m}=\frac{1}{m}\tan\theta=\frac{1}{m}\tan\alpha m=\alpha.\frac{\tan\alpha m}{\alpha m},$$
hence, in the limit, we have $\dfrac{t^2}{m^2}=\alpha^2$, and
$$\frac{\tan\alpha}{\alpha}=\frac{1}{1-}\,\frac{\alpha^2}{3-}\,\frac{\alpha^2}{5-}\,\frac{\alpha^2}{7-}\,\frac{\alpha^2}{9-}\cdots.$$

This series was used by Lambert to prove the irrationality of π, since such a C.F. is irrational.

J. M. CHILD.

SOME INCIDENTAL WRITINGS BY DE MORGAN.

WHILE gathering the materials for something in the nature of a "Companion" to De Morgan's "Budget of Paradoxes," I came across a considerable number of his notes, short papers, and other contributions to the famous English periodical known as "Notes and Queries." The first number appeared on Nov. 3, 1849, and from that date "N. & Q." has been a household word among those who are afflicted with the insatiable desire to penetrate into the bye-ways of Literature, Genealogy, Heraldry, Antiquarian Lore, the Science of History and the History of Science—all formed grist for those who came to that mill. By the kind permission of the Editor of this invaluable publication, I have been permitted to reproduce from its columns what is presented to the reader in the following pages. The early numbers of "N. & Q." are exceedingly scarce, and a complete set is hardly to be obtained by those who live at a distance from the great libraries.

Those who find the following extracts amusing and instructive will perhaps bear in mind how useful a medium "N. & Q." is for enquiries in almost every field of literature and science. It has an advantage over the *Gazette* in at least one respect—it is a weekly, and answers may reach enquirers after a much shorter interval than is possible in our case. But there is no need to dwell further on the merits of "N. & Q." What is important, is that the Editor of that periodical should know that the readers of the *Gazette*, as well as the writer of those lines, are grateful for his instant and generous consent to a request for permission to extract such communications as it received from De Morgan in the first decade or so of its existence.

The communications to "N. & Q." are for the most part signed "M." or with the name in full. There are a few which from internal evidence may be attributed to De Morgan, and I have found one in which identity is concealed under a numerical signature.

They are not all on mathematical subjects. They include replies to questions on the variety of topics likely to be raised in such a periodical as "N. & Q." But they are almost all on matters of common interest, and reveal the wide culture and omnivorous reading of one to whom books were as the salt of life. De Morgan did not always "suffer fools gladly," and his excursions into the field of dialectic are characterised by much pretty play with the rapier, with occasionally an exhibition of the skilful use of the bludgeon or quarter-staff. At any rate, in so far as they show the play of an alert and vigorous intellect, and a devotion to ethical principle, they are too good to be lost. I have borne in mind the needs of the younger reader, and where names, books, and references now unfamiliar to the average person are introduced I have ventured, as far as my own reading and distance from large libraries permitted, to add many notes, titles of books referred to, and quotations of interest, especially from other writings of De Morgan. It is hoped that no apology is needed for this. Many of the names of men are those of half-forgotten worthies, who in their day played their part gallantly enough, and by no means deserve the total or partial oblivion with which they are threatened.

All students of the History of Mathematics should possess themselves of the remarkable Catalogues issued by Messrs. Sotheran under the title, ***Bibliotheca Chemico-Mathematica.*** They are of great intrinsic interest, and I believe them to be deserving of almost implicit confidence. Hereinafter I refer to them as "S." They are worthy of a firm which has been in existence for over a century.

The context will sufficiently indicate the notes occasionally added to queries, etc., by the Editor of "N. & Q." This was the antiquary, William John Thoms (1803-1885), deputy librarian to the House of Lords. He founded "N. & Q." in the year 1849.

D. N. B. refers to the *Dictionary of National Biography*; *N. E. D.* to the great *Oxford Dictionary*.

All material not directly from "N. & Q." is placed in square brackets.

I. i. 281 is De Morgan's own proposed notation for Series I. volume i. p. 281.

Additional notes, corrections, and suggestions will be welcomed by the compiler. One must cover almost the whole field of knowledge to do De Morgan full justice.

1849. I. i. 107. **Boswell's Arithmetic.**—[This query is headed "Dr. Johnson and Professor de Morgan," and is answered in the course of De Morgan's reply to another query, I. viii. 198.]—Mr. Editor,—Although your cleverly conceived publication may be considered as more applicable to men of letters than to men of figures, yet I doubt not you will entertain the subject I am about to propound: because, in the first place, "whole generations of men of letters" are implicated in the criticisms; and, in the next place, because however great, as a man of figures, the critic may be, the man of letters criticised was assuredly greater.

Professor de Morgan has discovered a flaw in the great Johnson! and, in obedience to your epigraph, "*when found make a note of it,*" he *has* made a note of it at the foot of p. 7 of *The Companion to the Almanac* for 1850,—eccola:—

"The following will show that a palpable absurdity will pass before the eyes of *generations of men of letters* without notice. In Boswell's *Life of Johnson* (chapter viii. of the edition with chapters), there is given a conversation between Dr. Adams and Johnson, in which the latter asserts that he could finish his Dictionary in three years.

"*Adams.* 'But the French Academy, which consists of forty members, took forty years to compile their Dictionary.' *Johnson.* 'Sir, thus it is. This is the proportion. Let me see: forty times forty is sixteen hundred, so is the proportion of an Englishman to a Frenchman.'

"No one of the numerous editors of Boswell has *made a note upon this*, although many things as slight have been commented upon: it was certainly not Johnson's mistake, for he was a clear-headed arithmetician. How many of our readers will stare and wonder what we are talking about, and what the mistake is!"

Certes, I for one, plead guilty to staring, and wondering what the Professor is talking about.

I cannot for a moment imagine it possible, that he could base such a criticism, so announced, upon no better foundation than the mere verbal transposition of the words Englishman and Frenchman.

The inversion deceives no person, and it is almost more appropriate to the colloquial jocularity of the great Lexicographer's bombast than if the enunciation had been more strictly according to rule. Besides, the correctness of the expression, even as it stands, is capable of defence. Let the third and fourth terms be understood as referring to *time* instead of to *power*, and the proportion becomes "as three to sixteen hundred so is" (the time required by) "an Englishman to" (that required for the same work by) "a Frenchman."

Or, if natives be referred to in the plural,—then, as three to sixteen hundred, so are Englishmen to Frenchmen; that is, such is the number of each required for the same amount of work.

But I repeat that I cannot conceive a criticism so trifling and questionable can have been the true aim of Professor de Morgan's note, and as I am unable to discover any other flaw in the Doctor's proportion, according to the premises, *my query*, Mr. Editor, has for its object to learn

" *What the mistake is?* " B.

[What Boswell really gave was: "Let me see; forty times forty is sixteen hundred. As three to sixteen hundred, so is the proportion of an Englishman to a Frenchman," *v.* Vol. I. c. v. p. 99 (Routledge, edn. in 4 vols.). For another "jolly confusion of ideas," from "Tristram Shandy," v. 3, cf. *The Budget of Paradoxes* (hereinafter referred to as *The Budget*), pp. 417–418 (1872).]

I. viii. 198. A. E. B. quoting from the *Companion to the Almanac* for 1850, p. 9, cites De Morgan's remark, explaining the usage of attainment of majority:

"Nevertheless in the law, which here preserves *the old reckoning*, he is of full age on the 9th: though he were born on the 10th, he is of age to execute a settlement *a minute after midnight* on the morning of the 9th." This statement clashes with that in the opening scene of Ben Jonson's *Staple of News*, where Pennyboy jun. counts as his watch strikes one, two, three ... six!

"Enough, enough, dear watch,
Thy pulse hath beat enough
—The hour is come so long expected," &c.

Then the "fashioner" comes in to fit on the heir's new clothes; he had waited below 'till the clock struck, and gives, as an excuse, "your worship might have pleaded *nonage*, if you had got 'em ere I could make just affidavit of the time."

A. E. B. asks how the two statements can be reconciled, seeing that these "particulars are too *verbatim* to admit of doubt as to the peculiar usage of that time."

De Morgan replies, p. 250:

Attainment of Majority.—A. E. B. has not quoted quite correctly. He has put two phrases of mine into Italics, which makes them appear to have special relation to one another, while the word which *I* put in Italics, "*ninth*," he has made to be "9th." Farther, he has left out some words. The latter part should run thus, the words left out being in brackets:

"... though he were born [a minute before midnight] on the 10th, he is of age to execute a settlement at a minute after midnight on the morning of the 9th, forty-eight hours all but two minutes before he has drawn breath for the space of twenty-one years."

Had the quotation been correct, it would have been better seen that I no more make the day of majority begin a minute after midnight, than I make the day of birth end a minute before midnight. A second, or even the tenth of a second, would have done as well.

The *old reckoning*, of which I was speaking, was the reckoning which rejects fractions; and the matter in question was the *day*. For my illustration, any beginning of the day would have done as well as any other; on this I must refer to the paper itself. Nevertheless, I was correct in implying that the day by which age is reckoned begins at midnight; and I believe it began at midnight in the time of Ben Jonson. The law recognised two kinds of days;—the natural day of twenty-four hours, the artificial day from sunrise to sunset. The birthday, and with it the day of majority, would needs be the natural day; for otherwise a child not born by daylight would have no birthday at all. I cannot make out that the law ever recognised a day of twenty-

four hours beginning at any hour except midnight. For payment of rent, the artificial day was recognised, and the tenant was required to tender at such time before sunset as would leave the landlord time to count the money by daylight; a reasonable provision, when we think upon the vast number of different coins which were legal tender. But even here it seems to have been held that though the landlord might enter at sunset, the forfeiture could not be enforced if the rent were paid before midnight. A legal friend suggested to me that perhaps Ben Jonson had more experience of the terminus of the day as between landlord and tenant, than of that which emancipates a minor. This would not have struck me: but a lawyer views man simply as the agent or patient in distress, ejectment, *quo warranto*, &c.

A. E. B. twice makes the question refer to *usage*, whereas I was describing *law*. If I were as well up in the drama as I should like to be, I might perhaps find a modern plot which turns upon a minor coming of age, in which the first day of majority is what is commonly called the *birthday*, instead of, as it ought to be, the day before. Writers of fiction have in all times had fictitious law. If we took decisions from the novelists of our own day, we should learn, among other things, that married women can in all circumstances make valid wills, and that the destruction of the parchment and ink which compose the material of a deed is also the destruction of all power to claim under it.

Singularly enough, this is the second case in which my paper on reckoning has been both misquoted and misapprehended in "N. & Q." My knowledge of the existence of this periodical began with a copy of No. 7 (containing p. 107, vol. i.), forwarded to me by courtesy of the Editor, on account of a Query signed (not A. E. B. but) B., affirming that I had "discovered a flaw in the great Johnson!" Now it happened that the flaw was described, even in B.'s own quotation from me, as "certainly not Johnson's mistake, for he was a clear-headed arithmetician." B. gave me half a year to answer; and then, no answer appearing, privately forwarded the printed Query, with a request to know whether the readers of "N. & Q." were not of a class sufficiently intelligent to appreciate a defence from me. The fact was, that I thought them too intelligent to need it, after the correction (by B. himself, in p. 127) of the misquotation. It is not in letters as in law, that judgment must be signed for the plaintiff if the defendant do not appear. There is also an anonymous octavo tract, mostly directed, or at least (so far as I have read) much directed, against the arguments of the same article, and containing misapprehensions of a similar kind. That my unfortunate article should be so misunderstood in three distinct quarters, is, I am afraid, sufficient presumption against its clearness; and shows me that *obscurus fio* is, as much as ever, the attendant of *brevis esse laboro*: but I am still fully persuaded of the truth of the conclusions. A. De Morgan.

On p. 296 of the same volume A. E. B. quotes from "a black letter octavo" entitled *A Concordancie of Yeares*, published in 1615 (and for that year), chap. xiii.:

"The day is of two sorts, natural and artificiall: the natural day is the space of 24 hours, in which time the sunne is carried by the first Mover, from the east into the west, and so round about the world into the east againe."

"The artificiall day continues from sunne-rising to sunne-setting: and the artificiall night is from the sunne's setting to his rising. And you must note that this natural day, according to divers, hath divers beginnings: As the Romanes count it from mid-night to mid-night, because at that time our Lorde was borne, being Sunday; and so do we account it for fasting dayes. The Arabians begin their day at noone, and end at noone the next day; for because they say the sunne was made in the meridian; and so do all astro-

nomers account the day, because it alwayes falleth at one certaine time. The Umbrians, the Tuscans, the Jewes, the Athenians, Italians, and Egyptians, do begin their day at sunne-set, and so do we celebrate festivall dayes. The Babylonians, Persians, and Bohemians begin their day at sunne-rising, holding till sunne-setting ; *and so do our lawyers count it in England.*"

The author quoted is Arthur Hopton, "a distinguished mathematician, a scholar of Oxford, a student in the Temple" ; and the volume itself is dedicated to "The Right Honourable Sir Edward Coke, Knight, Lord Chiefe Justice of England," &c.

Mr. Russell Gole writes (p. 371) : The misunderstanding which has arisen between Professor De Morgan and A. E. B. has proceeded, it appears, from the misapplication of the statement of the latter's authority (Arthur Hopton) to the question at issue. Where Hopton says that our lawyers count their day from sunrise to sunset, he, I am of opinion, merely refers to certain instances, such as distress for rent :

"A man cannot distrain for rent or rent-charge in the night (which, according to the author of *The Mirror*, is after sunset and before sunrising)."—*Impey on Distress and Replevin*, p. 49.

In common law, the day is now supposed among lawyers to be from six in the morning to seven at night for service of notices ; in Chancery, till eight at night. And a service after such times at night would be counted as good only for the next day. In the case of Liffin *v.* Pitcher, 1 *Dowl. N.S.* 767., Justice Coleridge said, "I am in the habit of giving twenty-four hours to plead when I give one day." Thus it will be perceived that a lawyer's day is of different lengths.

With regard to the time at which a person arrives at majority, we have good authority in support of Professor De Morgan's statement :

"So that full age in male or female is twenty-one years, which age is completed on the day preceding the anniversary of a person's birth, who till that time is an infant, and so styled in law."—Blackstone's *Commentaries*, vol. i. p. 463.

There is no doubt also that the law rejects fractions of a day where it is possible :

"It is clear that the law rejecteth all fractions of days for the uncertainty, and commonly allows him that hath part of the day in law to have the whole day, unless where it, by fraction or relation, may be a prejudice to a third person."—Sir O. Bridgm. 1.

And in respect to the present case it is quite clear. In the case of Reg. *v.* the Parish of St. Mary, Warwick, reported in the *Jurist* (vol. xvii. p. 551), Lord Campbell said :

"In some cases the Court does not regard the fraction of a day. Where the question is on what day a person came of age, the fraction of the day on which he was born and on which he came of age is not considered."

And farther on he says :

"It is a general maxim that the law does not regard the fraction of a day."

Prof. De Morgan sums up the discussion at this point (p. 372) : I only treat misquotation as an *offence* in the old sense of the word ; and courteously, but most positively, I deny the right of any one who quotes to omit, or to alter emphasis, without stating what he has done. That A. E. B. did misunderstand me, I was justified in inferring from his implication that I made the day begin "a minute after midnight."

Arthur Hopton, whom A. E. B. quotes against me (but the quotation is from chapter xiv., not xiii.), is wrong in his law. The lawyers,

from Coke down to our own time, give both days, the natural and artificial, as legal days. See Coke Littleton (Index, *Day*), the current commentators on Blackstone, and the usual law dictionaries.

Nevertheless, this discussion will serve the purpose. No one denies that the day of majority now begins at midnight: no one pretends to prove, by evidence of decisions, or opinions of writers on law, that it began otherwise in 1600. How then did Ben Jonson make it begin, as clearly as A. E. B. shows he does, at six o'clock (meaning probably a certain sunrise)? Hopton throws out the natural day altogether in a work on chronology, and lays down the artificial day as the only one known to lawyers: it is not wonderful that Jonson should have fallen into the same mistake. A. DE MORGAN.

[An elaborate reply by A. E. B. will be found on pp. 571-3 of the same volume, after which the subject seems to have dropped.]

I. x. 363. **Boswell's Arithmetic.**—I once pointed out a mistake which Boswell had fixed on Johnson. The curiosity is, not that Boswell should have blundered, but that so many editors should have allowed the blunder to pass. I now point out another such mistake, and submit it for correction.

"*Boswell.* I wish to have a good walled garden.

"*Johnson.* I don't think it would be worth the expense to you. We compute, in England, a park wall at a thousand pounds a mile; now a garden wall must cost at least as much. You intend your trees should grow higher than a deer will leap. Now let us see; for a hundred pounds you could only have forty-four square yards, which is very little; for two hundred pounds you may have eighty-four [eighty-eight of course] square yards, which is very well."—*Boswell's Johnson*, aetat. 74, vol. viii. p. 195 of Croker's ten-volume edition.

On this there is one commentator, according to Mr. Croker, namely, the Bishop of Ferns (Dr. Elvington, the editor of *Euclid*, I suppose). The Bishop says that Boswell makes Johnson talk nonsense, and that it ought to be fourty-four *yards square* instead of fourty-four *square yards*. This makes the matter worse. I think I see how the confusion arose in Boswell's mind, but at present I leave it as a Query.

A. DE MORGAN.

(*To be continued.*)

MATHEMATICAL NOTES.

508. [K[1]. 8. a.] *Brocard Points for a Quadrilateral.*

If a point X can be found within a cyclic quadrilateral $ABCD$, such that the angles XAD, XBA, XCB, XDC are equal, then

$$BC.AD = DC.AB.$$

Let $ABCD$ be the quadrilateral; denote the sides BC, CD, DA, AB by a, b, c, d; the diagonals DB, AC by e, f; the area by Q; the circum-radius by R and each of the angles XAD, XBA, XCB, XDC by ω. Then

$$\angle AXB = \pi - \omega - (A - \omega) = \pi - A.$$

Similarly $\quad \angle BXC = \pi - B,\ \angle CXD = \pi - C,\ \angle DXA = \pi - D.$

Now $$\frac{AX}{\sin\omega} = \frac{d}{\sin AXB} = \frac{d}{\sin(\pi - A)} = \frac{d}{\sin A}$$

and $$\frac{AX}{\sin(D-\omega)} = \frac{c}{\sin AXD} = \frac{c}{\sin(\pi - D)} = \frac{c}{\sin D};$$

$$\therefore \frac{\sin(D-\omega)}{\sin\omega} = \frac{d\sin D}{c\sin A}, \text{ or } \cot\omega = \frac{d}{c}\operatorname{cosec} A + \cot D.$$

Similarly $\qquad \cot\omega = \frac{c}{b}\operatorname{cosec} D + \cot C$, etc.;

hence,

$$\cot\omega = \frac{d}{c}\operatorname{cosec} A + \cot D = \frac{c}{b}\operatorname{cosec} D + \cot C = \frac{b}{a}\operatorname{cosec} C + \cot B = \frac{a}{d}\operatorname{cosec} B + \cot A$$

or $\quad \frac{d}{c}\operatorname{cosec} A - \frac{a}{d}\operatorname{cosec} B + \frac{b}{a}\operatorname{cosec} C - \frac{c}{b}\operatorname{cosec} D = \cot A - \cot B + \cot C - \cot D,$

the general condition for all quadrilaterals.

When the quadrilateral is cyclic, $\operatorname{cosec} A = \operatorname{cosec} C$, $\cot A = -\cot C$, etc.; therefore

$$\left(\frac{d}{c} + \frac{b}{a}\right)\operatorname{cosec} A = \left(\frac{c}{b} + \frac{a}{d}\right)\operatorname{cosec} B.$$

But $$(ab + cd)\sin A = (ad + bc)\sin B = 2Q;$$

hence $$ac = bd.$$

Simple Exercise. Show that $2\cot\omega = \dfrac{(ad+bc)(ab+cd)}{2acQ}$

$$= \frac{16R^2Q}{e^2f^2},$$

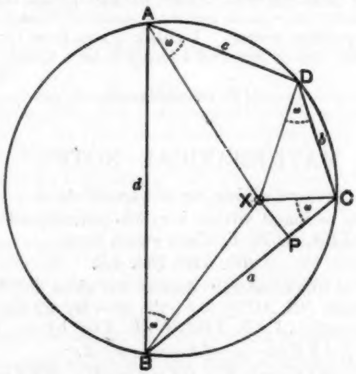
when the quadrilateral is cyclic.

Further, if X' be a point such that

$$\angle X'AB = \angle X'BC = \angle X'CD = \angle X'DA = \omega',$$

then, it may similarly be shown that $ac = bd$ and $\cot\omega' = \dfrac{8R^2Q}{e^2f^2}$; therefore $\omega' = \omega$.

Geometrically, X may be found as the common point of intersection of the circles described on AB, BC, CD, DA touching the sides AD, AB, BC, CD respectively, whilst X' is the common point of intersection of the circles described on AB, BC, CD, DA touching the sides BC, CD, DA, AB respectively.



To calculate the position of X, let P be the foot of the perpendicular from X on BC; let $CP = x$, $PX = y$, then

$$x = CX\cos\omega = \frac{b\sin\omega\cos\omega}{\sin C} = \frac{b(ab+cd)\cot\omega}{2Q(1+\cot^2\omega)} = \frac{bR}{e}.\sin 2\omega.$$

and $$y = x\tan\omega = \frac{b(ab+cd}{2Q(1+\cot^2\omega)} = \frac{2bR}{e}.\sin^2\omega.$$

Similarly, if $x'=P'B$, $y'=P'X'$, where P' is the foot of the perpendicular on BC from X', then

$$x'=\frac{d(ad+bc)\cot\omega}{2Q(1+\cot^2\omega)} \quad\text{and}\quad y'=\frac{d(ad+bc)}{2Q(1+\cot^2\omega)}.$$

Again, $\qquad XX'^2=XB^2+X'B^2-2\,.\,XB\,.\,X'B\,.\cos XBX'.$

Remembering that

$$\sin B=\frac{ab+cd}{2bd}\tan\omega=\frac{a^2+d^2}{2ad}\,.\tan\omega,$$

since $ac=bd$, and that $ef=2ac=2bd$, it is readily found that

$$XX'=\frac{efbd\cos\omega\,.\sqrt{\cos 2\omega}}{2RQ}=\frac{b^2d^2\cos\omega\,.\sqrt{\cos 2\omega}}{RQ}.$$

Particular Cases. Three will suffice.

(1) When $a=b=c=d$, the fundamental condition is satisfied, and $\cot\omega=1$ or $\omega=45°$. $x=y=\frac{1}{2}a$ and $XX'=0$, *i.e.* the points X, X' are coincident at the centre of the square.

It may readily be shown that 45° is the maximum value of ω.

(2) When $a=c$, $b=d$, the fundamental condition is not satisfied unless $a=b$, as in (1); thus no points can be found for a rectangle.

(3) When $a=c$, then d must be equal to $\dfrac{a^2}{b}$ for the points X, X' to be found; hence $e=f=a\sqrt{2}$ and CD is parallel to AB; therefore $B+C=\pi$. Hence

$$\angle BXC+\angle CXD=\pi-B+\pi-C=\pi\,;$$

therefore BX, XD are segments of the diagonal BD. Further,

$$\frac{BX}{XD}=\frac{a\sin\omega}{\sin B}\cdot\frac{\sin D}{c\sin\omega}=1\,;$$

therefore X is the mid-point of BD.

Similarly X' is the mid-point of AC. F. G. W. Brown.

509. [K. 8. a, 2. a.] *Points, Lines and Circles connected with the Complete Quadrilateral.*

I have tried in this Note to collect the most interesting theorems dealing with points, lines and circles connected with the complete quadrilateral, In order to do so I have made a rather extensive, though not exhaustive. search after the facts, and have added two or three properties which I have not found stated elsewhere.

This does not profess to be a complete list of theorems relating to the quadrilateral. I have omitted all references to conic sections (except twice in footnotes), all theorems which hold only for special cases such as inscribed or circumscribed quadrilaterals, all metrical theorems and a few others which might have been inserted, but which seemed to be of less importance than the fifteen that are given.

I. *The Wallace point.* Let ABC be a triangle, cut by a transversal PQR, so that A, P; B, Q; C, R are the opposite vertices of a complete quadrilateral. Let its diagonals AP, BQ intersect at W; BQ, CR at U; CR, AP at V. Let the middle points of the diagonals be X, Y, Z. The four sides of the quadrilateral determine four triangles ABC, AQR, BRP, CPQ.

1°. The circumcircles of these four triangles are concurrent[1] at a point,

[1] Three or more circles will be called *concurrent* if they have *one* point in common, *coaxal* if *two* points in common.

referred to in this paper as the *Wallace point* of the quadrilateral.[1] Denote it by O.

2°. The points where the tangent at A to the circle ABC cuts QR, where the tangent at A to the circle AQR cuts BC, where the tangent at P to the circle PCQ cuts BR, where the tangent at P to the circle PBR cuts CQ are concyclic, and the points A and P lie on this circle. Two other circles are similarly connected with the points B, Q and C, R. The three circles are concurrent at the Wallace point.[2]

3°. Let L_1, L_2 be the Wallace points of the quadrilaterals AU, BC, QR; PU, BR, CQ. Then the circle passing through A and the middle points of BR and CQ also passes through L_1 and O; and the circle passing through P and the middle points of BC and QR also passes through L_2 and O. Similarly if M_1, M_2, N_1, N_2 are the Wallace points of the quadrilaterals BV, CA, RP; QV, CP, AR; CW, AB, PQ; RW, AQ, BP, four other circles may be drawn which concur at O with the two circles already mentioned, and which pass through M_1, M_2, N_1, N_2 respectively.

Again, circles determined by U, mid BQ, mid CR and two similar circles, pass through L_1, L_2; M_1, M_2; N_1, N_2 respectively.[3]

Also, circles L_1AO, L_2PO, L_1L_2U are concurrent; similarly the other circles are concurrent in threes.

Finally, circles L_1L_2U, M_1M_2V, N_1N_2W are concurrent, and the circle UVW passes through their point of concurrence.[4]

Fourteen circles pass through the Wallace point, four in 1°, three in 2°, six in 3°, and one to be mentioned in 4°.

II. *The circumcentric circle.* Call the centres of the circles circumscribing the triangles ABC, AQR, BRP, CPQ, O_1, O_2, O_3, O_4, respectively.

4°. The points O_1, O_2, O_3, O_4, O are concyclic on a circle which will be called the *circumcentric circle* of the quadrilateral.[5]

5°. AO_1, QO_4, RO_3 intersect at a point X_2, the intersection of the circumcentric circle with the circle centre O_2, and there are three other such points determined by three other such sets of lines.[6]

6°. Sixteen circles are determined by the points A, B, C, P, Q, R taken in threes, excluding the four cases where three of the points are collinear. At each of the six vertices of the quadrilateral two circles meet, each determined by three of these sixteen centres,—by one O_1, O_2, O_3 or O_4, with two of the twelve new centres. The twelve circles obtained in this way are divided into three sets of four circles each, of which two circles of each set meet at one vertex and the other two at the opposite vertex. Moreover, the four circles of each set concur at a point on the circumcentric circle.

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[1] This theorem was first stated by "Scoticus," probably a pseudonym for Dr. William Wallace (Mackay, *Proc. Edin. Math. Soc.*, 9, 1890-91), in 1804, in Leybourn's *Math. Repos.*, vol. 1, part 1, p. 170. It was restated by Steiner in 1828 as the first of ten theorems proposed for solution in Gergonne's *Annales*, vol. 18.

[2] Casey, *Analytical Geometry*, p. 535, Ex. 81. It is easily proved by pure geometry.

[3] Carles states that L_1, L_2, U, Y, Z are concyclic (*Jour. de Math. élém.*, 1883).

[4] In fact they concur at the Wallace point of the quadrilateral UX, VY, WZ. Cf. 7°.

[5] This is the second of Steiner's theorems (Gergonne's *Annales*, vol. 18, 1828). It was restated in 1835 by T. S. Davies of Bath, and proved by him in Leybourn's *Mathematical Repository*, Q. 555, vol. 6. Probably Davies discovered the theorem independently of Steiner; that Davies had been studying the quadrilateral for some time is evidenced by his statement in connection with the solution of Q. 524 in the *Repository*, and by Mackay's remark (*Proc. Edin. Math. Soc.*, vol. 9. 1890-91) that Davies had proposed a question in 1821 in the Leeds *Correspondent* dealing with a property of this figure. At any rate he published the first proof of the theorem. After proving the theorem, Davies proceeds with several other properties of a quadrilateral, including a special case of 5°.

[6] Hermes, *Nouv. Annales de Math.*, 1859, Q. 476. Solved, p. 359.

Twelve points then lie on the circumcentric[1] circle, five of 4°, four of 5°, three of 6°.

III. *The orthocentric line and the mid-diagonal line.*

7°. Circles described on the three diagonals of the quadrilateral as diameters are coaxal. Their line of centres will be called the *mid-diagonal line.*[2]

8°. The circle circumscribing the triangle UVW, and the polar circles[3] of the triangles ABC, AQR, BRP, CPQ form a coaxal system orthogonal to the coaxal system of 7°. The line of centres of this system is called the *orthocentric line.*[4]

9°. From the feet of the perpendiculars from A, B, C to PQR lines are drawn respectively perpendicular to BC, CA, AB. These three lines are concurrent at a point on the orthocentric line. The point is called the *orthopole* (Neuberg) of the transversal PQR with respect to the triangle ABC. Similarly three other points can be found.[5]

Nine remarkable points lie on the orthocentric line,—the four orthocentres, the four orthopoles and the centre of the circle UVW. Three remarkable points lie on the mid-diagonal line, the mid-points of the diagonals.[6]

IV. *The pedal line.*

10°. The feet of the perpendiculars let fall from the Wallace point on the four sides of the quadrilateral are collinear in a line which will be called the *pedal line* of the quadrilateral. The pedal line and the orthocentric line are parallel, and the pedal line passes through the middle point of the perpendicular drawn from the Wallace point to the orthocentric line.[7]

11°. Circles drawn on OA, OB, OR as diameters are concurrent at a point on the pedal line, the foot of the perpendicular from O to ABR. Similarly three other sets of three circles are concurrent at three other points on the pedal line.[8]

12°. The nine-point circle of ABC cuts the pedal line in two points. Through one of these points pass the nine-point circles of the triangles OAB, OBC, OCA.[9] Through the other passes the pedal, Simson or Wallace line

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[1] Hermes calls this circle the eight-point circle (*loc. cit.*). Gallatly calls it the centre circle (*Mod. Geom. of Tri.*, p. 5).

[2] J. T. Connor proved in 1795, in the *Ladies' Diary*, that the middle points of the diagonals of the complete quadrilateral are collinear. This was Steiner's third theorem (Gergonne, *loc. cit.*). Steiner attributed the theorem to Newton, but there is no evidence that it was known prior to 1795. 7° was stated in the *Repository*, Q. 555, by Davies, and again in the same number as Q. 600 by Twaddleton.

[3] The polar circle of a triangle is a circle, having the orthocentre for centre, and with respect to which the triangle is self-conjugate. The circle is imaginary unless the triangle is obtuse angled.

[4] Steiner stated that the orthocentres of the four triangles are collinear on a line perpendicular to the mid-diagonal line (Gergonne, *loc. cit.*). Davies proved the fact in the *Repository* (*loc. cit.*). The earliest reference I have found to the complete theorem 8° is in a paper by Mention, to be referred to later (*Nouv. Annales de Math.*, vol. 1, p. 16), where its truth is taken for granted as if it were well known. For a proof see Durell, *Plane Geometry for Advanced Students*, part 1, Theorem 86, p. 188 (1909).

It is well known that the parabola which touches the four sides of the quadrilateral has O for its focus, and the orthocentric line for its directrix.

[5] See Gallatly, *Mod. Geom. of Triangle*, Chap. VI.

[6] The mean centres of the vertices of the quadrilaterals $BCQR$, $CARP$, $ABPQ$ also lie on the mid-diagonal line. One of the parabolas determined by the centroids of the four triangles ABC, AQR, BRP, CPQ has its axis parallel to the mid-diagonal line, and is so placed that the Wallace point is two-thirds as far from the mid-diagonal line as from the axis.

[7] This is due to Steiner (Gergonne, *loc. cit.*); proofs were first given by Davies (*Repos.*, Q. 555, *loc. cit.*).

[8] Catalan, *Théorèmes et Problèmes*, 6th edit., p. 34.

[9] *The Lady's and Gentleman's Diary*, 1864, p. 55.

of the point on the circle circumscribing ABC directly opposite O. This line is perpendicular to the pedal line.[1] Similarly for the points in which the nine-point circles of AQR, BRP, CPR cut the pedal line.

There are twelve points on the pedal line,—the feet of the perpendiculars from O to the sides of the quadrilateral, and the eight points in which the line is cut by the four nine-point circles.

V. *Steiner's lines and circles.* Denote the internal bisectors of the angles BAC, ABC, BCA, BPR, CQR, BRQ by the letters α, β, γ, π, ρ, σ, and the external bisectors of the same angles by the letters α', β', γ', π', ρ', σ'; denote the point of intersection of α and π by $\alpha\pi$, and so on.

13°. For each of the four triangles ABC, AQR, BRP, CPQ there are an inscribed and three escribed circles, making in all sixteen circles, whose centres are four by four concyclic so as to give rise to eight new circles, which will be referred to as *Steiner's circles.*

These eight circles are divided into two groups such that each of the four circles of one group cuts all the circles of the other group orthogonally; whence it follows that the centres of the circles of the two groups are on two straight lines perpendicular to each other; these lines will be called *Steiner's lines.*

Steiner's lines intersect at the Wallace point.[2]

14°. The points $\alpha\pi$, $\alpha'\pi'$; $\beta\rho$, $\beta'\rho'$; $\gamma\sigma$, $\gamma'\sigma'$ form the vertices of a complete quadrilateral; the points $\alpha\pi'$, $\alpha'\pi$; $\beta\rho'$, $\beta'\rho$; $\gamma\sigma'$, $\gamma'\sigma$ form the vertices of a second complete quadrilateral. These may be called *Mention's quadrilaterals.* The orthocentric line of either of these quadrilaterals is the mid-diagonal line of the other. The two orthocentric lines are perpendicular to each other and are the Steiner's lines of the original quadrilateral.[3]

It follows that the three circles determined by A, P, $\alpha\pi$, $\alpha'\pi'$; B, Q, $\beta\rho$, $\beta'\rho'$; C, R, $\gamma\sigma$, $\gamma'\sigma'$ are coaxal with the circles of one of Steiner's group of circles; and that the three circles determined by A, P, $\alpha\pi'$, $\alpha'\pi$; and two other such sets of points are coaxal with the other group.[4] Also the circles circumscribing the diagonal triangles of Mention's quadrilaterals belong one to each of these coaxal systems.

15°. The diagonal triangles of the original triangle and of Mention's quadrilaterals are similar and are also in perspective, the centre of perspective being the orthocentre of the diagonal triangle of the original quadrilateral.[4]

All of the theorems of this paper may be quite simply proved by Euclidean methods, illustrating Lachlan's remark that "it might well be taken as an axiom, based upon experience, that every geometrical theorem admits of a simple and direct proof by the principles of Pure Geometry."[5] Any of the theorems mentioned above might have been discovered by the Greek geometers; but as far as we know the best of them had to wait two thousand years for the insight of Steiner to perceive them. Probably other beautiful relations in connection with the quadrilateral lie buried, waiting to be brought to light by accident, by laborious digging or by the use of a powerful analytical tool in a hand strong enough to wield it.

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[1] Casey, *Sequel to Euclid*, p. 164, Ex. 138 (1886).

[2] This beautiful and surprising theorem is the last and best of Steiner's theorems (Gergonne, *loc. cit.*). Like his other theorems on the quadrilateral, it was never proved in the *Annales*, which ceased publication in 1831. In 1862, Mention (*Nouv. Annales de Math.*, vol. 1, 1862, pp. 16 and 65), making use of several new theorems, gave an elaborate proof of 13°. As Mention points out, it is probable that Steiner possessed a simpler proof which he never published.

[3] Mention, *Nouv. Annales*, 1862, p. 16.

[4] Sancery, *Nouv. Annales*, vol. 14, p. 145, 1875.

[5] *Modern Pure Geometry*, Preface, edit. of 1893.

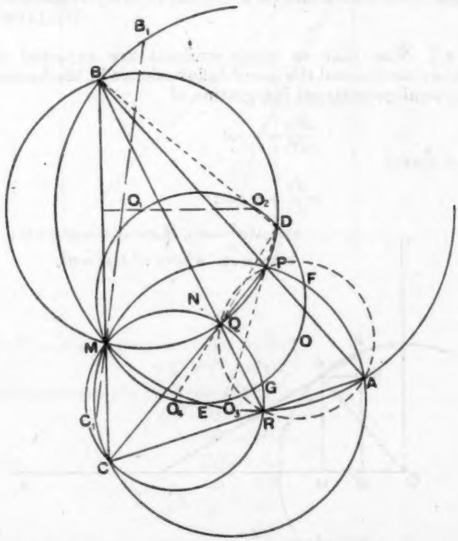
510. [K. 2. e.] (Vol. vii. p. 107, No. 392.) This "remarkable circle" is not only "a nine-point circle," but is actually the nine-point circle itself. Instead of beginning with four lines, however, it is more instructive to begin at the opposite end of the argument by considering first two circles.

1. Let O_1 and O_2 be the centres of two circles intersecting at B and M. Join one of these common points, say B, to the centres, and let BO_1 cut O_2 in E, and BO_2 cut O_1 at D; then M, O_1, O_2, D, E, all lie on a circle N, for the angles $O_1MO_2 = O_1EO_2 = O_1DO_2 = O_1BO_2$; hence the five points are on N, and BO_1E, BO_2D are by construction straight lines.

2. Let D be the second common point of O_1 and N, where O_1 is on the circumference of N; then any line through D cutting N and O_1 cuts off a segment $O_2B = O_2M$, for $O_1BO_2 = O_1DO_2 = O_1MO_2$, but $O_1B = O_1M$; hence $O_2B = O_2M$. So also $O_3P = O_3M$ and $O_4Q = O_4M$.

3. The circle O_3 cuts O_1, O_2 at P and A collinear with B. By our first applied to O_2O_3, we have AO_3E a straight line. The supplement of $O_3AP = APO_3 + AO_3P = EMB$, but $EMD = AO_3P$; hence $APO_3 = DMB = DPB$, and O_3PD being a straight line BPA is a straight line.

4. Applying this to circles $O_1O_2O_4$ gives BQR a straight line, while $O_1O_3O_4$ and $O_2O_3O_4$ give CQP, CRA straight lines, which together with BPA form a quadrilateral the centres of whose triangle circumcircles lie on a circle, while they all pass through a common point M on this circle and also cut it again at D, E, F, G, while through the common point of N and each of the circumcircles lines pass through the other three centres and the vertices of the triangle circumscribed; thus for D we have DO_2B, DO_3P, DO_4Q, and similar groups for E, F, and G.



5. When the join of O_3O_4 is parallel to the join of O_1O_2, the quadrangle $APQR$ is cyclic, for these two lines are at right angles to MC and MB; hence BMC is a straight line, and the angle $APC = AMC$ and $ARB = AMB$. The centre of the quadrangle circle O is also on the centre circle, for $O_1O_3O_4 = A = O_1OO_4$.

6. Take B_1MC_1 at right angles to AM, cutting O_2, O_3 at B_1, C_1; then AO_2B_1, AO_3C_1 are diameters and O_2, O_3 are the mid points of AB_1 and AC_1, and B_1EA, C_1FA are right angles; hence E and F are the feet of the perpendiculars from B_1, C_1 on the sides AC_1 and B_1A, that is, N is the nine-point circle of AB_1C_1.

7. When the quadrangle is cyclic the following points may be noted:

(1) O is always on the perpendicular to BC at M.

(2) O_1 and O_4 are on the perpendiculars from B and C to the opposite sides.

(3) O_2 and O_3 are on perpendiculars to AB and AC through S the circumscribed centre.

(4) The locus of M is the side BC.

(5) The locus of Q is the circle BHC, H the orthocentre.

(6) O_1, O_2, O_3, O_4 belong to four co-axial systems of circles, whose radical axes are parallel in pairs.

(7) Circles $PQRA$ belong to a co-axial system, whose radical axis is the median AA'.

(8) Other two triangles, one having B, the other C as fixed points, can be found for which N is nine-point circle.

(9) The orthocentre of ABC is the orthocentre of the triangle formed by joining the orthocentres of the three triangles which have N as their nine-point circle, S is the circumcentre of the triangle formed from their three circumcentres, N is the nine-point centre of the triangle formed by their nine-point centres, and G is the centroid of the triangle formed by their three centroids; and these four triangles are equal in every respect.

WILLIAM FINLAYSON.

511. [C. 2. a.] Now that so many students are expected to learn the elements of the Calculus and the use of infinitesimals in Mechanical Problems, the following semi-geometrical integration of

$$\frac{d^2x}{dt^2} = -\mu x$$

may be found useful,

$$v\frac{dv}{dx} = -\mu x;$$

$$\therefore\ v^2 = \mu(a^2 - x^2),\ \text{if } x = a \text{ when } v = 0;$$

$$\therefore\ v = \sqrt{\mu}\,.\,y,\ \text{where } x^2 + y^2 = a^2.$$

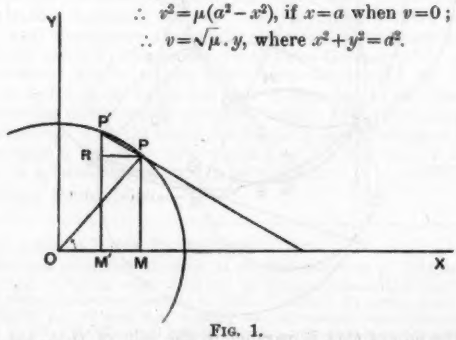


FIG. 1.

With obvious geometrical construction, MP, $M'P'$ being two consecutive ordinates and dt the interval of time, and PR perpr. to $M'P'$.

$$PP' = a\,d\theta, \quad PR = v\,dt;$$

$$\therefore \frac{d\theta}{dt} = \frac{a}{v} \cdot \frac{PP'}{PR}$$
$$= \frac{a}{v} \cdot \frac{MP}{OP} \text{ in the limit}$$
$$= \sqrt{\mu};$$
$$\therefore \theta = \sqrt{\mu} . t, \text{ if } t=0 \text{ when } x=a.$$

The more advanced student may be interested in seeing how a similar method applies to the equation

$$\frac{d^2x}{dt^2} = \mu x$$

with the use of the rectangular hyperbola $x^2 - y^2 = a^2$ instead of the circle, the only properties required being that in the limit $OY . OP = a^2$,

$$\angle MPT = \angle MOP.$$
$$v^2 = \mu(x^2 - a^2),$$
$$v = \sqrt{\mu} . y \text{ when } x^2 - y^2 = a^2.$$

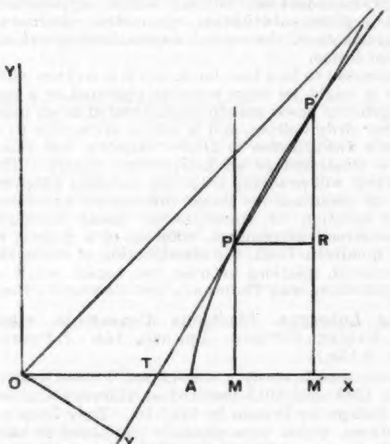


FIG. 2.

Then if OY be drawn perpendicular to PP'

$$OY . PP' = a^2 du,$$

using Levett's notation $a^2u = 2 \text{ Area } AOP$,

$$OY . OP = a^2,$$
$$PP' = OP\, du,$$

and as before,

$$PR = v\, dt;$$
$$\therefore \frac{v}{OP} \cdot \frac{dt}{du} = \frac{PR}{PP'}$$
$$= \frac{MP}{OP} \text{ in the limit};$$
$$\therefore du = \sqrt{\mu} . dt,$$
$$u = \sqrt{\mu} . t, \text{ if } t=0 \text{ when } x=a,$$
$$x = a \cosh \sqrt{\mu} . t.$$

E. M. LANGLEY.

REVIEWS.

Homogeneous Linear Substitutions. By HAROLD HILTON. Pp. 184. 12s. 6d. net. 1914. (Oxford : Clarendon Press.)

Of late years the theory of groups of linear substitutions has been especially cultivated, and, in the second edition (1911) of his *Theory of Groups of Finite Order*, Prof. W. Burnside has given a fairly long account of it, and has even said that "for further advances in the abstract theory [of groups] one must look largely to the representation of a group as a group of linear substitutions." In the book under review, Prof. Hilton has put together those properties of the homogeneous linear substitution with real or complex coefficients, of which frequent use is made in the theory of groups and in the theory of bilinear forms and invariant-factors. Only those properties of substitutions are dealt with which do not depend on any "group" of such substitutions. The main reason for this limitation is the fact that the most interesting properties of groups of substitutions may be found in the work of Prof. Burnside referred to above.

The chapters of the book under review are on elementary properties of substitutions, invariant-factors, bilinear forms, applications, substitutions permutable with a given substitution, symmetric, alternate, and Hermitian substitutions, invariants of the second degree, orthogonal substitutions, and families of bilinear forms.

This work is intended to be a text-book, but it is written with such condensed competence that it might be more suitably regarded as a monograph on the subject of homogeneous linear substitutions treated as an independent theory. The book is rather drily written, and is not so attractive to a student as, for example, Bôcher's *Introduction to Higher Algebra*, but this is probably inseparable from a treatment as an independent theory. Thus, the greatest interest to a student will probably lie in the excellent chapter on applications to the solution of simultaneous linear differential equations with constant coefficients, the solution of simultaneous linear homogenous equations, collineation, geometrical movements, solution of a quartic equation, critical values of a real quadratic form, the classification of conicoids, the tangential equation of a conicoid, relations between two conics, small oscillations about a position of equilibrium, and Thomson's and Bertrand's theorems.

Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire. By C. DE LA VALLÉE POUSSIN. Pp. viii, 154. 7 francs. 1916. (Paris : Gauthier-Villars et Cie.)

The author, who was formerly a well-known Professor at the University of Louvain, and in 1914 and 1915 lectured at Harvard University, gave these lectures at the Collège de France in 1915-16. They form a continuation of the Harvard lectures, which were partially published in the *Transactions of the American Mathematical Society* for 1915, and the book under review is included in the collection of monographs on the theory of functions edited by Émile Borel.

In 1898 Borel gave a definition of the "measure" of aggregates, which differed in some important respects from the definitions in vogue before him, and which were ultimately based on Hankel's definition of "content" and Riemann's theory of integration. To the ideas thus set in motion by Borel belong the famous researches of Lebesgue on integration and on the general notion of an "additive function of an aggregate." Further, Lebesgue made important progress (1910) in the theory of functions representable by infinite series of functions which are either continuous or point-wise discontinuous, which was inaugurated by Baire (1899).

In this book the first part is on measurable aggregates and Lebesgue's integral; the second part is on additive functions of an aggregate, and the author deals with the question more precisely than he did in the Harvard lectures, and introduces new methods. Thus, among such additive functions the most important are "absolutely continuous" functions, which are the same things as the indefinite integrals of Lebesgue; and the derivation of these functions is studied by a new method which is a great improvement

on that used in the Harvard lectures. The third part is devoted to the transfinitely many classes into which Baire and Lebesgue have divided "analytically representable" functions.

This book is of the greatest value for those who are interested in the important branch of the theory of functions which is principally due to recent French writers. PHILIP E. B. JOURDAIN.

A Shilling Arithmetic. By JOHN W. ROBERTSON, M.A., B.Sc. Pp. 191 +vi. 1s. 1917. (G. Bell & Sons.)

The chief feature of this little book is its splendid collection of examples. According to the preface, they are practically all original, and the author is to be congratulated on his achievement in this respect. *Whitaker, The Statesman's Almanac,* everyday problems of war and finance, sugar and wheat imports and so on, have all in their turn supplied inspiration, while care has been taken to insert a sufficient number of "drill" examples which "are so essential for the average pupil." The captious might take offence at the old-fashioned treatment of logarithms, the position of graphs in the book, the omission of brackets in the examples on multiplication and division of fractions (*e.g.* what answer is required to Ex. 81 on p. 49 ?), and one or two other details; but the good points of the book on the whole are so obvious and make it such an excellent shillings-worth, that those who are thinking of introducing a new Arithmetic into their classes next September can be confidently recommended to inspect a copy before making their final choice. There is a slight misprint on p. 75. H. G. M.

A First Course in Projective Geometry. By E. HOWARD SMART, M.A. Pp. xxiii, 1-273. 7s. 6d. 1916. (Macmillan.)

The object of this book is in the first instance to serve as a first course in projection, and secondly, by means of the methods of projection and cross-ratio to introduce the student to the chief properties of the conic before proceeding to the study of the more advanced works on modern pure geometry. It may be said at once that the author has succeeded in the task which he set himself to perform, and has produced a book which, though dealing with such well-worn subjects, is distinctly original in its treatment and eminently readable.

In Chapter II. we are introduced to the elementary notions of projection and the ideas of correspondence and duality, the relations between copolar and coaxial triangles being proved in a very clear and simple manner. Chapters III. and IV. deal with the metrical theorems of Ceva and Menelaus, harmonic section and the harmonic properties of the complete quadrangle and quadrilateral. In Chapters V. and VI. the author leaves the straight line and treats of inversion, centres of similitude and coaxal circles, which naturally lead to poles and polars and the harmonic properties of the inscribed quadrangle and circumscribed quadrilateral, and attention is drawn to the principles of duality and reciprocation. Up to this point the author has been making a collection (with proofs) of the various geometrical properties which he intends to employ, and in Chapter VII. he again takes up projection and performs in full various standard projections, the figures and demonstrations being given in a clear and interesting manner, although we would suggest that fig. 42 on p. 83 would be improved by being drawn in perspective.

This may be said to constitute the first part of the book. In Chapter VIII. the author proceeds to deduce various properties of the conic from the corresponding properties of the circle by means of the method of projection. At the outset he defines a conic as the projection of a circle, any vertex and plane of projection being taken, and he shows that the conic, like the circle, is a curve of the second order and second class. He then shows how the position of the vanishing line in the plane of the circle discriminates between the parabola, ellipse and hyperbola, and introduces at once the ideas of poles and polars, conjugate points and lines, and from them obtains various properties relating to the centre, axes, conjugate diameters and asymptotes. In Chapter IX., from a consideration of Carnot's theorem, which is obtained by projection from the circle, he obtains various fundamental properties, *e.g.* the constant values of the ratios $PN^2 : AN \cdot NA'$ and $QV^2 : PV \cdot VP'$, and asymptotic pro-

perties of the hyperbola. In connection with these latter we would suggest that the author should reconsider the second proof given on p. 135, that the triangle CLL' is of constant area. The fact that the triangles LTM, $L'TM'$ are *ultimately* equal does not allow us to draw the conclusion that the triangles CLL', CMM' are equal, and therefore to this extent the proof is incomplete. A simple proof based on the equality of the intercepts on a secant between the curve and the asymptotes is as follows: Let LL' be the tangent at P. In any direction draw three parallel lines LM', NPN', ML'; let MM' and NN' meet at Q. Then $LP=PL'$, $M'N'=N'L'$, $QN'=\frac{1}{2}ML'=PN$, therefore Q lies on the curve. Also $M'Q=QM$, therefore MM' is the tangent at Q. The triangles CLL', CMM' are clearly equal, and all tangents may be dealt with by varying the direction of the parallels.

In Chapter X., from a consideration of pairs of conjugate lines at right angles, we are led to the foci and directrices and various properties connected with them, and Chapter XI. deals with the most important focal, tangent and normal properties, concluding with the theorems of Gaskin and Steiner and the determination of chords of curvature. In Chapters XII. and XIII. our author takes up the elementary properties of cross-ratio and homographic ranges and pencils, and by means of them obtains projective theorems on the conic, *e.g.* Chasles' anharmonic properties of the points and tangents of a conic, the locus ad tres et quatuor lineas, etc. The remaining chapters deal with the Pascal and Brianchon theorems, self-corresponding elements, the construction of a conic from given conditions and reciprocation. It will be seen that a considerable ground is covered, and as no previous knowledge is required, save that of Euclid I.-XI., the book may justly be described as an excellent introduction to modern pure geometry.

A good feature of the book, and one which certainly adds to its interest, is the historical notes which are given at the ends of most of the chapters, and these we think might be extended with advantage, *e.g.* it might be stated that the theorem on p. 27, and there attributed to Desargues, is the only one of Euclid's porisms which has come down to us in a complete form. On p. 125 it would be as well to state that the "so-called Newton's theorem" is given by Apollonius (see the author's note on p. 159), in view of the fact that it is subsequently referred to, on p. 132, as "Newton's theorem" without qualification. The second paragraph of the Historical note on p. 225 should be re-written. It was pointed out by a writer in the *Math. Gazette*, No. 6, of October, 1895, that the first solution of the locus ad tres et quatuor lineas was given by Apollonius in Bk. III., Props. 17, 19, 21, 23, as indeed was definitely stated by him in the general introduction to his *Conics*. How this solution was overlooked for 2000 years by mathematicians, even by men like Descartes and Newton, who, in all probability, used his *Conics* as their text-book, may well be called an insoluble riddle. On p. 235 it might be added that Pascal enunciated (without proof) his theorem for the circle only.

The following errata have been noted in the text, and should be corrected in the next edition:

P. 56, line 4, for "one and one" read "one and only."

P. 69, in fig. 33b, for "O" read "C."

P. 144, line 19, for "$K(KASA')$" read "$K(XASA')$." JOHN J. MILNE.

The Elements of Engineering Drawing. By E. ROWARTH. Pp. xii+131. 2s. 6d. net. 1916. (Messrs. Methuen.)

This is a thoroughly practical manual introductory to the subject, calculated to familiarise the young student at school or in the technical classes of institutes with the use of the instruments which they will be called upon at a later stage to employ to some purpose. The author claims that the conscientious working out of the examples in accordance with his instruction will free students from the reproach of the critics that they possess that "drawing-office style" which "has but a remote resemblance to that of the professional draughtsman, so frequently the deplorable result of training the student exclusively by the use of models. The book is the outcome of the experience gained by the author as Instructor in Engineering Drawing and Design in the Department of Mechanical and Civil Engineering at the Battersea Polytechnic, and elsewhere.

Oxford University Press Catalogue. Issued November, 1916, by Humphrey Milford, Publisher to the University of Oxford. Pp. viii + 566.

The first 480 pages of this book consist of a Subject Catalogue, and the rest of an Alphabetical List, with prices and references to the preceding pages. Supplements are to be issued containing similar information as to all books published after November, 1916. These will be sent to regular customers or supplied on application. Section V., headed "Natural Science and Medicine," consists of thirty-two pages, of which four contain all the mathematical works published by this Press. Taken as a whole, and quite apart from its ostensible object, the book lends itself to occasional browsing, and is rendered attractive by the inclusion of many typical illustrations from works in the lists.

Hydraulic Flow Reviewed. A Book of Reference of Standard Experiments on Pipes, Channels, Notches, Weirs, and Circular Orifices, together with New Formulae relating thereto. By A. A. Barnes. Pp. viii + 158. 12s. 6d. net. 1916. (Messrs. Spon.)

This valuable contribution to the literature of practical hydraulics lies somewhat outside the scope of the *Gazette*. It brings out clearly the wide divergence between theoretical results and the records of experiment. This is due to "the various unknown influences at work, and the best we can do is to make accurate experiments which shall, when a sufficient number has been accumulated, be allowed to supersede in practice all theoretical *assumptions*." The complexity of the subject is such that laboratory results are too often misleading, and no results are really trustworthy which are not derived from the comparison of the results obtained in real life and on a very large scale, and in sufficient number.

Compendio de Álgebra de Abenbéder. Texto árabe, traducción y estadia por J. A. S. Pérez. Pp. xlviii + 117 + the Arabic text. 6 ptas. 1916. (Junta Para Amplicación de Estudios, Moreto, 1, Madrid.)

The influence of the young and growing body of mathematical thought in Spain is making itself felt, and the publication of a translation of an Arabic Algebra from the MSS. preserved in the Library of the Escurial, is sufficient evidence of the vigour of the new school, and of the enterprise of a body of historians who take the history of science as part of their province. The library description of this interesting relic of the past is as follows:

CMXXXI. Codex literis cuphicis exaratus, quo continentur.

> i. Tractatus tripartitus, exaratus die 11 Schevali, anno Egirae 744 Christi 1343 ubi de Logistica, Apologistica & Analogistica disseritur, hac inscriptione: Algebrae et Comparationum Epitome: Hujus auctor Abi Abdalla Mohamad ben Omar, vulgo Ben Badr Hispalensis, egregius quidem, sed incertae aetatis scriptor. . . ."

The first part is devoted to theory, and is divided into seventeen sections: I.-VI., equations of the first and second degree; VII.-XII., operations with roots of numbers; XIII. and XIII. bis, XIV. and XIV. bis, multiplication of signs, with problems thereupon; XV., division; XVI. continues the second part of XIII.; XVII. is untranslatable: *Cheber y almocabala*. As the word algebra is derived from *alchéber*, the Arabic books on the subject had as title: *Libro de chéber y almocábala*, these being Arabic words signifying the operations necessary for the solution of equations. *Chéber* is the series of operations necessary to bring all the unknowns on to one side of the equations and to simplify them; while *almocábala* is the name for the operations necessary from that stage to find the value of the unknowns. So that to apply *el chéber y el almocábala* is to solve an equation. There is nothing in the treatise about the properties of progressions, although the examples which constitute the second and practical part do imply a knowledge of the relations between the first and last terms, the sum, the common difference, and the number of terms of an arithmetical progression. We are not competent to speak of the accuracy of the translation from the Arabic, and any reader of the *Gazette* is welcome to test it at his leisure if he will apply to the Editor for a copy of the book. But we have noticed nothing in the Spanish that is not quite intelligible, so that with this reservation we may compliment Signor Pérez on having done a valuable piece of work.

Mathematical Problem Papers. Compiled and arranged by the Rev. E. M. RADFORD. Pp. vi+203. 4s. 6d. net. 1914. (Cambridge University Press.)

Solutions to Mathematical Problem Papers. By E. M. RADFORD. Pp. vi+560. 10s. 6d. net. 1915. (Cambridge University Press.)

In happier days these volumes would have demanded an earlier and fuller notice than we have been able to give them. It will be remembered that the compiler published a large number of solutions of his problems some years ago in the *Mathematical Gazette*. No greater tribute to their value to teachers could be paid than is implied in the fact that the issues containing those solutions have long been out of print. The new edition of the *Problems* became necessary, partly because errors in the given results had in many cases to be corrected, and wider class experience showed that changes here and there in the grading of the questions were desirable. The second section of the book is now enriched by questions on the Integral and Differential Calculus. Both volumes are unusually free from misprints. The position of D, No. 10, p. 35, is not stated, and in No. 6 on the next page those of D, E, F are not given. In problems such as the latter, it might be well to add "test results with reference to coincidence of G with any other remarkable point for which similar results are known, a form of check which is too often neglected. In No. 3, p. 92, for x_1^2 in the first numerator read x_1''. The slips in the *Solutions* are few and far between and of little moment, *e.g.* p. 16, line two, omit the comma; p. 24, line 1, for rods, read roots; No. 7, p. 51, for time, read line. We have run over a very large number of the solutions, and find many of them elegant, and nearly all of them carefully set out in steps within the compass of the ordinary student. For the scholarship candidate, and for work at the University, the papers will be found invaluable, while the hard-pressed teacher and the solitary student alike will, though for different reasons, find the book of solutions indispensable.

THE LIBRARY.

CHANGE OF ADDRESS.

THE Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

SCARCE BACK NUMBERS.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them:

Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

ERRATA.

Vol. viii. p. 248, Note 464, *for* "D. z. d." *read* "D. 2. d."

Vol. viii. p. 328, l. 18 up, *for* "he" *read* "Nicholson."

Vol. ix. p. 10, l. 15 up, *for* 0·616 *read* 0·247.

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